1(3): The Lagrange Derivative in Plane Polar Coordinates

ference: Google "convective derivative plane polar coordinates"
www.pleasemakeanote.blogspot.co.uk
nd cross check with Wolfram site.

This derivative is an important generalization of dynamics, in which the velocity is defined as a function of time and position. For the cylindrical coordinates:

$$\underline{v}(r,\theta,z,t) = v_r(r,\theta,z,t)\,\underline{e}_r + v_\theta(r,\theta,z,t)\,\underline{e}_\theta + v_z(r,\theta,z,t)\,\underline{k} \quad -(1)$$

and:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \frac{dr}{dt}\frac{\partial \underline{v}}{\partial r} + \frac{d\theta}{dt}\frac{\partial \underline{v}}{\partial \theta} + \frac{dz}{dt}\frac{\partial \underline{v}}{\partial z} \quad -(2)$$

using the chain rule of differentiation. In the time derivative the partial derivative is computed at a fixed position and the unit vectors are fixed in time so:

$$\frac{\partial \underline{v}}{\partial t} = \frac{\partial v_r}{\partial t}\underline{e}_r + \frac{\partial v_\theta}{\partial t}\underline{e}_\theta + \frac{\partial v_z}{\partial t}\underline{k} \quad -(3)$$

The other terms in eq. (2) are worked out as follows:

$$\frac{dr}{dt}\frac{\partial \underline{v}}{\partial r} = v_r\frac{\partial \underline{v}}{\partial r} = v_r\left(\frac{\partial}{\partial r}\left(v_r\underline{e}_r + v_\theta\underline{e}_\theta + v_z\underline{k}\right)\right) \quad -(4)$$

The Leibnitz Theorem is used as follows:

$$\frac{\partial}{\partial r}\left(v_r\underline{e}_r\right) = \frac{\partial v_r}{\partial r}\underline{e}_r + v_r\frac{\partial \underline{e}_r}{\partial r} \quad -(5)$$

$$\frac{\partial}{\partial r}\left(v_\theta\underline{e}_\theta\right) = \frac{\partial v_\theta}{\partial r}\underline{e}_\theta + v_\theta\frac{\partial \underline{e}_\theta}{\partial r} \quad -(6)$$

$$\frac{\partial}{\partial r}\left(v_z\underline{k}\right) = \frac{\partial v_z}{\partial r}\underline{k} + v_z\frac{\partial \underline{k}}{\partial r} \quad -(7)$$

) Using:

$$\frac{\partial \underline{e}_r}{\partial r} = \frac{\partial \underline{e}_\theta}{\partial r} = \frac{\partial \underline{k}}{\partial r} = \underline{0} \quad -(8)$$

then:

$$v_r \frac{\partial \underline{v}_r}{\partial r} = v_r \left(\frac{\partial v_r}{\partial r} \underline{e}_r + \frac{\partial v_\theta}{\partial r} \underline{e}_\theta + \frac{\partial v_z}{\partial r} \underline{k} \right) -(9)$$

Now consider:

$$\frac{d\theta}{dt} \frac{\partial \underline{v}}{\partial \theta} = \frac{v_\theta}{r} \left(\frac{\partial}{\partial \theta} \left(v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{k} \right) \right) -(10)$$

because:

$$v_\theta = r\dot{\theta} = r \frac{d\theta}{dt} \quad -(11)$$

Using the Leibnitz Theorem:

$$\frac{\partial}{\partial \theta}\left(v_r \underline{e}_r\right) = \frac{\partial v_r}{\partial \theta} \underline{e}_r + v_r \frac{\partial \underline{e}_r}{\partial \theta} \quad -(12)$$

$$\frac{\partial}{\partial \theta}\left(v_\theta \underline{e}_\theta\right) = \frac{\partial v_\theta}{\partial \theta} \underline{e}_\theta + v_\theta \frac{\partial \underline{e}_\theta}{\partial \theta} \quad -(13)$$

$$\frac{\partial}{\partial \theta}\left(v_z \underline{k}\right) = \frac{\partial v_z}{\partial \theta} \underline{k} + v_z \frac{\partial \underline{k}}{\partial \theta} \quad -(14)$$

and:

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \,; \quad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \,; \quad \frac{\partial \underline{k}}{\partial \theta} = \underline{0} \quad -(15)$$

so

$$\frac{d\theta}{dt} \frac{\partial \underline{v}}{\partial \theta} = \frac{v_\theta}{r} \left(\left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) \underline{e}_r + \left(v_r + \frac{\partial v_\theta}{\partial \theta} \right) \underline{e}_\theta + \frac{\partial v_z}{\partial \theta} \underline{k} \quad -(16)\right.$$

Finally:

$$\frac{dZ}{dt}\frac{\partial \underline{v}}{\partial Z} = \frac{dZ}{dt}\left(\frac{\partial}{\partial Z}\left(v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_Z \underline{k}\right)\right) \quad -(17)$$

Using the Leibnitz Theorem:

$$\frac{\partial}{\partial Z}\left(v_r \underline{e}_r\right) = \frac{\partial v_r}{\partial Z}\underline{e}_r + v_r \frac{\partial \underline{e}_r}{\partial Z} \quad -(18)$$

$$\frac{\partial}{\partial Z}\left(v_\theta \underline{e}_\theta\right) = \frac{\partial v_\theta}{\partial Z}\underline{e}_\theta + v_\theta \frac{\partial \underline{e}_\theta}{\partial Z} \quad -(19)$$

$$\frac{\partial}{\partial Z}\left(v_Z \underline{k}\right) = \frac{\partial v_Z}{\partial Z}\underline{k} + v_Z \frac{\partial \underline{k}}{\partial Z} \quad -(20)$$

in which:

$$\frac{\partial \underline{e}_\theta}{\partial Z} = \frac{\partial \underline{e}_r}{\partial Z} = \frac{\partial \underline{k}}{\partial Z} = \underline{0} \quad -(21)$$

so:

$$\frac{dZ}{dt}\frac{\partial \underline{v}}{\partial Z} = v_Z\left(\frac{\partial v_r}{\partial Z}\underline{e}_r + \frac{\partial v_\theta}{\partial Z}\underline{e}_\theta + \frac{\partial v_Z}{\partial Z}\underline{k}\right) - (22)$$

Adding eqs. (9), (16) and (22):

$$\begin{aligned}\frac{D\underline{v}}{Dt} = &\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_Z\frac{\partial v_r}{\partial Z}\right)\underline{e}_r \\ &+ \left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_Z\frac{\partial v_\theta}{\partial Z}\right)\underline{e}_\theta \\ &+ \left(\frac{\partial v_Z}{\partial t} + v_r\frac{\partial v_Z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_Z}{\partial \theta} + v_Z\frac{\partial v_Z}{\partial Z}\right)\underline{k}\end{aligned} \quad -(23)$$

This is the same as the Wolfram result, Q.E.D

For plane polar coordinates:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}$$

$$= \frac{\partial v_r}{\partial t}\underline{e}_r + \frac{\partial v_\theta}{\partial t}\underline{e}_\theta \qquad -(\overset{24}{\cancel{17}})$$

$$+ \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} \right)\underline{e}_r$$

$$+ \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} \right)\underline{e}_\theta$$

Now use:

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta} \qquad -(\overset{25}{\cancel{18}})$$

so:

$$v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = \dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} - r\dot{\theta}^2 \qquad -(\overset{26}{\cancel{19}})$$

and

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} \qquad -(\overset{27}{\cancel{20}})$$

$$= \dot{r}\frac{\partial (r\dot{\theta})}{\partial r} + \dot{\theta}\frac{\partial (r\dot{\theta})}{\partial \theta} + \dot{r}\dot{\theta}$$

Now use the Leibnitz theorem to find that:

$$\frac{\partial (r\dot{\theta})}{\partial r} = \dot{\theta} + r\frac{\partial \dot{\theta}}{\partial r}, \qquad -(\overset{28}{\cancel{21}})$$

$$\frac{\partial (r\dot{\theta})}{\partial \theta} = \dot{\theta}\frac{\partial r}{\partial \theta} + r\frac{\partial \dot{\theta}}{\partial \theta} \qquad -(\overset{29}{\cancel{22}})$$

Therefore the acceleration in general is a convective acceleration. This realization introduces new terms into planar orbital theory.

Therefore:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \left(\dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} - r\dot{\theta}^2 \right)\underline{e}_r$$
$$+ \left(2\dot{r}\dot{\theta} + r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + r\dot{\theta}\frac{\partial \dot{\theta}}{\partial \theta} + \dot{\theta}^2 \frac{\partial r}{\partial \theta} \right)\underline{e}_\theta \qquad -(\overset{30}{\cancel{23}})$$

where we have used:

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r}$$
$$= \dot{r}\frac{\partial (r\dot{\theta})}{\partial r} + \dot{\theta}\frac{\partial (r\dot{\theta})}{\partial \theta} + \dot{r}\dot{\theta}$$
$$= \dot{r}\left(\dot{\theta} + r\frac{\partial \dot{\theta}}{\partial r}\right) + \dot{\theta}\left(\dot{\theta}\frac{\partial r}{\partial \theta} + r\frac{\partial \dot{\theta}}{\partial \theta}\right) + \dot{r}\dot{\theta}$$
$$= 2\dot{r}\dot{\theta} + r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + r\dot{\theta}\frac{\partial \dot{\theta}}{\partial \theta} + \dot{\theta}^2\frac{\partial r}{\partial \theta} \qquad -(\overset{31}{\cancel{24}})$$

Q.E.D.

Now note that:

$$\frac{\partial \dot{\theta}}{\partial \theta} = \frac{\partial \dot{\theta}}{\partial t}\frac{\partial t}{\partial \theta} = \frac{\ddot{\theta}}{\dot{\theta}} \qquad -(\overset{32}{\cancel{25}})$$

so:

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + \left(\dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} - r\dot{\theta}^2 \right)\underline{e}_r$$
$$+ \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2\frac{\partial r}{\partial \theta} \right)\underline{e}_\theta \qquad -(\overset{33}{\cancel{25}})$$

Note carefully that the usual expression for

) acceleration in plane polar coordinates is:

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} - r\dot{\theta}^2 \underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(\cancel{26}\ 34)$$

Comparing eqs. ($\cancel{25}$ 33) and ($\cancel{26}$ 34) it is seen that there is the additional acceleration term:

$$\underline{a}_1 = \left(\dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta}\right)\underline{e}_r + \left(r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2\frac{\partial r}{\partial \theta}\right)\underline{e}_\theta \quad -(\cancel{27}\ 35)$$

when the Lagrange derivative is used.

The new fundamental accelerations in eq. ($\cancel{27}$ 35) occur in addition to the centrifugal, ~~cent~~ Coriolis and other accelerations of eq. ($\cancel{26}$ 34). The accelerations of eq. ($\cancel{27}$ 35) are presumably considered in the subject of fluid dynamics. Following the unification of fluid dynamics with gravitation, they should also be considered in the subject of gravitation, and in general dynamics, and should be looked for experimentally.

The new accelerations ($\cancel{27}$ 35) originate in the use of the velocity field (1), in which $\underline{v}$ is a function of coordinates as well as time. The derivation of eq. ($\cancel{26}$ 34) follows from the fact that the position of a particle is defined with respect to a given reference frame

) by a vector $\underline{r}$, which is considered to be a function of time t. So the velocity and acceleration are defined by:

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} \quad - (\overset{36}{\cancel{28}})$$

and

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} \quad - (\overset{37}{\cancel{29}})$$

Therefore for this definition:

$$(\underline{v} \cdot \underline{\nabla})\, \underline{v} = \underline{0} \quad - (\overset{38}{\cancel{30}})$$

and

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} \quad - (39)$$

and from Note 361(2):

$$\begin{bmatrix} \omega^1{}_{01} & \omega^1{}_{02} & \omega^1{}_{03} \\ \omega^2{}_{01} & \omega^2{}_{02} & \omega^2{}_{03} \\ \omega^3{}_{01} & \omega^3{}_{02} & \omega^3{}_{03} \end{bmatrix} = \begin{bmatrix} \partial v_X/\partial X & \partial v_X/\partial Y & \partial v_X/\partial Z \\ \partial v_Y/\partial X & \partial v_Y/\partial Y & \partial v_Y/\partial Z \\ \partial v_Z/\partial X & \partial v_Z/\partial Y & \partial v_Z/\partial Z \end{bmatrix} \quad - (\overset{40}{\cancel{32}})$$

$$= 0$$

The spin connections vanish and there is no dependence of velocity on position. This is the usual Newtonian or inertial definition. The definition of $\underline{r}$ in Cartesian coordinates is

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad - (\overset{41}{\cancel{35}})$$

where $\underline{i}$, $\underline{j}$ and $\underline{k}$ do not depend on time. Therefore:

8)

$$\underline{r}(t) = X(t)\,\underline{i} + Y(t)\,\underline{j} + Z(t)\,\underline{k} \quad -\left(\overset{42}{\cancel{34}}\right)$$

In plane polar coordinates

$$\underline{r} = r\,\underline{e}_r \quad -\left(\overset{43}{\cancel{35}}\right)$$

where

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -\left(\overset{44}{\cancel{36}}\right)$$

where

$$\theta = \theta(t) \quad -\left(\overset{45}{\cancel{37}}\right)$$

Therefore the unit vector $\underline{e}_r$ is time dependent in the plane polar system. The other unit vector is:

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -\left(\overset{46}{\cancel{38}}\right)$$

It follows that:

$$\dot{\underline{e}}_r = (-\sin\theta)\,\dot\theta\,\underline{i} + (\cos\theta)\,\dot\theta\,\underline{j} = \left(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right)\dot\theta = \dot\theta\,\underline{e}_\theta \quad -\left(\overset{47}{\cancel{39}}\right)$$

$$\dot{\underline{e}}_\theta = (-\cos\theta)\,\dot\theta\,\underline{i} - (\sin\theta)\,\dot\theta\,\underline{j} = -\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}\right)\dot\theta = -\dot\theta\,\underline{e}_r \quad -\left(\overset{48}{\cancel{40}}\right)$$

i.e.

$$\dot{\underline{e}}_r = \dot\theta\,\underline{e}_\theta \quad -\left(\overset{49}{\cancel{41}}\right)$$

$$\dot{\underline{e}}_\theta = -\dot\theta\,\underline{e}_r \quad -\left(\overset{50}{\cancel{42}}\right)$$

These relations were used in derivation of eq. $\left(\overset{35}{\cancel{27}}\right)$. The position vector in plane polar coordinates is therefore found using:

$$X = r\cos\theta \quad -\left(\overset{51}{\cancel{43}}\right)$$

$$Y = r\sin\theta \quad -\left(\overset{52}{\cancel{44}}\right)$$

and

$$\underline{i} = \underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta \quad -\left(\overset{53}{\cancel{45}}\right)$$

$$\underline{j} = \underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta \quad -\left(\overset{54}{\cancel{46}}\right)$$

1) So:

$$\underline{r} = X\underline{i} + Y\underline{k} = (r\cos\theta)(\underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta) \qquad -\left(\overset{55}{\cancel{47}}\right)$$
$$+ (r\sin\theta)(\underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta)$$
$$= r\,\underline{e}_r$$

in which:

$$r = r(t), \quad \underline{e}_r = \underline{e}_r(t) \qquad -\left(\overset{56}{\cancel{48}}\right)$$

Therefore:

$$\underline{v} = \frac{d}{dt}(r\underline{e}_r) = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt}$$
$$= \frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta \qquad -\left(\overset{57}{\cancel{49}}\right)$$
$$= \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta$$

and

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \qquad -\left(\overset{58}{\cancel{50}}\right)$$
$$= \frac{dX}{dt}\underline{i} + \frac{dY}{dt}\underline{j} \qquad -\left(\overset{58a}{\cancel{50a}}\right)$$

Similarly, the acceleration is:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta) \qquad -(59)$$
$$= \ddot{r}\,\underline{e}_r + \dot{r}\,\dot{\underline{e}}_r + \frac{d}{dt}(r\dot{\theta})\underline{e}_\theta + r\dot{\theta}\,\dot{\underline{e}}_\theta$$
$$= \ddot{r}\,\underline{e}_r + \dot{r}\,\dot{\underline{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta + r\dot{\theta}\,\dot{\underline{e}}_\theta$$
$$= \ddot{r}\,\underline{e}_r + \dot{r}\dot{\theta}\,\underline{e}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta - r\dot{\theta}^2\underline{e}_r$$
$$= (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta$$

Q.E.D.

0) So:

$$\underline{a} = \frac{d^2x}{dt^2}\,\underline{i} + \frac{d^2y}{dt^2}\,\underline{j} \qquad -\overset{60}{(\cancel{52})}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_\theta$$

and the velocity and acceleration can be expressed either in a static Cartesian frame or moving plane polar frame.

The movement of the frame produces the non-Newtonian Coriolis accelerations first inferred by

It is seen that if:

$$\underline{a}_1 = \underline{0} \qquad -\overset{61}{(\cancel{53})}$$

then:

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} \qquad -\overset{62}{(\cancel{54})}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_\theta$$

in plane polar coordinates. Therefore the movement of the frame can be represented by:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = -r\dot{\theta}^2\,\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_\theta \qquad -\overset{63}{(\cancel{54})}$$

in the plane polar coordinate system, which assumes that eq. $\overset{61}{(\cancel{53})}$ is true, i.e. plane polar coordinates assume a spin connection such that eq. $\overset{62}{(\cancel{54})}$ is true.

In the general coordinate system acceleration in a plane is given by Eq. ($\cancel{25}$) 33

This conclusion affects the whole of orbital theory. The velocity used for eq. ($\cancel{25}$) 33 is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad - (\cancel{55})\ 64$$

so the lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{mMG}{r} \quad - (\cancel{56})\ 65$$

This is superficially similar to the usual lagrangian of two dimensional orbital theory, but from eq. (1) it is seen that r, $\dot{r}$ and $\dot{\theta}$ all depend on r, θ and t in fluid gravitation.

In the usual treatment the Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad - (\cancel{57})\ 66$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad - (\cancel{58})\ 67$$

and the angular momentum is constant:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \dot{\theta}. \quad - (\cancel{59})\ 68$$

In consequence:

$$\dot{\theta} = \frac{L}{mr^2} \quad - (\cancel{60})\ 69$$

and

$$2\dot{r}\dot{\theta} = 2\left(\frac{L}{mr^2}\right)^2 \frac{dr}{d\theta} \quad - \left(\overset{70}{\cancel{61}}\right)$$

Similarly:

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{L}{mr^2}\right) = \frac{d}{dr}\left(\frac{L}{mr^2}\right)\frac{dr}{dt}$$

$$= -\frac{2L}{mr^3}\frac{dr}{dt} = -\frac{2L^2}{m^2r^4}\frac{dr}{d\theta} \quad - \left(\overset{71}{\cancel{62}}\right)$$

Therefore for all orbits in a plane, plane polar coordinates produce:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad - \left(\overset{72}{\cancel{63}}\right)$$

so we obtain the Leibnitz equation:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r \quad - \left(\overset{73}{\cancel{64}}\right)$$

From eq. $\left(\overset{35}{\cancel{27}}\right)$, it is now known that the plane polar coordinate system uses the constraint equations

$$\dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} = 0 \quad - \left(\overset{74}{\cancel{65}}\right)$$

and

$$r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2\frac{\partial r}{\partial \theta} = 0 \quad - \left(\overset{75}{\cancel{66}}\right)$$

If these are not used, then new orbital features emerge.