Note 361(1): General Dynamics with the Lagrange Derivative

General dynamics is developed with the Euler equation:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v} \quad -(1)$$

where $\underline{a}$ is the acceleration. As shown in UFT351, eq. (1) is an example of the Cartan covariant derivative:

$$\frac{DV^a}{\partial x^\mu} = \frac{\partial V^a}{\partial x^\mu} + \omega^a_{\ \mu b} V^b \quad -(2)$$

where V^a is a vector in any dimension and $\omega^a_{\ \mu b}$ is the spin connection.

Therefore general dynamics is developed automatically into a theory of general relativity.

In Cartesian coordinates eq. (1) in three dimensions is:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \left(v_X \frac{\partial}{\partial X} + v_Y \frac{\partial}{\partial Y} + v_Z \frac{\partial}{\partial Z} \right) \underline{v} \quad -(3)$$

i.e.:

$$\frac{Dv_X}{Dt} = \frac{\partial v_X}{\partial t} + v_X \frac{\partial v_X}{\partial X} + v_Y \frac{\partial v_X}{\partial Y} + v_Z \frac{\partial v_X}{\partial Z} \quad -(4)$$

$$\frac{Dv_Y}{Dt} = \frac{\partial v_Y}{\partial t} + v_X \frac{\partial v_Y}{\partial X} + v_Y \frac{\partial v_Y}{\partial Y} + v_Z \frac{\partial v_Y}{\partial Z} \quad -(5)$$

$$\frac{Dv_Z}{Dt} = \frac{\partial v_Z}{\partial t} + v_X \frac{\partial v_Z}{\partial X} + v_Y \frac{\partial v_Z}{\partial Y} + v_Z \frac{\partial v_Z}{\partial Z} \quad -(6)$$

If we define:

$$\underline{V} = \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} \quad -(7)$$

$$-(8)$$

then:

$$\frac{D}{Dt}\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} + \begin{bmatrix} \partial V_X/\partial X & \partial V_X/\partial Y & \partial V_X/\partial Z \\ \partial V_Y/\partial X & \partial V_Y/\partial Y & \partial V_Y/\partial Z \\ \partial V_Z/\partial X & \partial V_Z/\partial Y & \partial V_Z/\partial Z \end{bmatrix}\begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix}$$

where the matrix is the Jacobian. It defines the spin connection matrix:

$$\begin{bmatrix} \omega^1{}_{01} & \omega^1{}_{02} & \omega^1{}_{03} \\ \omega^2{}_{01} & \omega^2{}_{02} & \omega^2{}_{03} \\ \omega^3{}_{01} & \omega^3{}_{02} & \omega^3{}_{03} \end{bmatrix} = \begin{bmatrix} \partial V_X/\partial X & \partial V_X/\partial Y & \partial V_X/\partial Z \\ \partial V_Y/\partial X & \partial V_Y/\partial Y & \partial V_Y/\partial Z \\ \partial V_Z/\partial X & \partial V_Z/\partial Y & \partial V_Z/\partial Z \end{bmatrix} \quad -(9)$$

From the ECE2 unification of gravitation and fluid dynamics it follows that the Newtonian acceleration due to gravity:

$$\underline{g} = \frac{d\underline{v}}{dt} = -\underline{\nabla}\phi = -\frac{mMG}{r^2} \quad -(10)$$

is generalized to:

$$\underline{g} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = -\underline{\nabla}\phi \quad -(11)$$

i.e. to an Euler equation.

For a planar orbit it follows that:

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial X} + v_Y \frac{\partial v_x}{\partial Y} \quad -(12)$$

$$\frac{Dv_Y}{Dt} = \frac{\partial v_Y}{\partial t} + v_x \frac{\partial v_Y}{\partial X} + v_Y \frac{\partial v_Y}{\partial Y} \quad -(13)$$

The non-Newtonian part of the acceleration $\underline{g}$ in eq. (11) is:

$$\underline{g}_1 = (\underline{v} \cdot \underline{\nabla})\underline{v} \quad -(14)$$

and the Newtonian or inertial part of the acceleration is

$$\underline{g}_N = \frac{\partial \underline{v}}{\partial t}, \quad -(15)$$

so the complete acceleration is:

$$\underline{g} = \underline{g}_N + \underline{g}_1 \quad -(16)$$

In immediately preceding papers it has been shown that the velocity:

$$\underline{v} = (MG)^{1/2} \frac{(-X\underline{i} + Y\underline{j})}{(X^2+Y^2)^{3/4}} \quad -(16)$$

used in eq. (14) produces:

$$\underline{g}_1 = -\frac{MG}{(X^2+Y^2)^{1/2}} \underline{e}_r \quad -(17)$$

and $\underline{v}$ was identified with the orbital velocity in a plane.

7) In ECE2 / Kambe fluid dynamics:

$$\underline{g}_1 = (\underline{v}\cdot\underline{\nabla})\underline{v} = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla}\phi \quad -(18)$$

where $\underline{v}$ is the orbital velocity. From eq. (16) it follows that:

$$v_X = -\frac{(MG)^{1/2}X}{(x^2+y^2)^{3/4}} \quad -(19)$$

and

$$v_Y = \frac{(MG)^{1/2}Y}{(x^2+y^2)^{3/4}} \quad -(20)$$

It is clear that the orbital velocity is produced by the non-Newtonian part of the acceleration, and that this is the Lagrange derivative of the orbital velocity. The Newtonian acceleration between m and M would result in m being attracted to M in a line joining m and M. The Newtonian acceleration alone does not produce an orbit.

For a circular orbit:

$$x^2 + y^2 = r^2 = \alpha^2 \quad -(21)$$

where α is the constant half right latitude. So for a circular orbit:

$$\frac{\partial v_X}{\partial X} = -\left(\frac{MG}{\alpha^3}\right)^{1/2}, \quad \frac{\partial v_X}{\partial Y} = 0, \quad -(22)$$

$$\frac{\partial v_Y}{\partial X} = 0, \quad \frac{\partial v_Y}{\partial Y} = \left(\frac{MG}{\alpha^3}\right)^{1/2} \quad -(23)$$

5) So the Jacobian matrix of spin connections is:

$$\begin{bmatrix} \omega^1{}_{01} & \omega^1{}_{02} \\ \omega^2{}_{01} & \omega^2{}_{02} \end{bmatrix} = \left(\frac{MG}{d^3}\right)^{1/2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(24)$$

The presence of a spin connection means that the theory is a theory of general relativity based on Cartan geometry. Without the spin connection there is no orbit.

For a circular orbit the non-Newtonian acceleration is:

$$\underline{g}_1 = -\frac{mMG}{r^2}\underline{e}_r = (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(25)$$

where

$$v^2 = \frac{MG}{r} \quad -(26)$$

is the square of the orbital velocity. For a circular orbit this is constant:

$$v^2 = \frac{MG}{d}. \quad -(27)$$

From the theory of fluid gravitation (ECE2/Kambe fluid dynamics):

$$\underline{g}_1 = -\frac{mMG}{r^2}\underline{e}_r = -\frac{\partial \underline{v}}{\partial t} - \underline{\nabla}\phi \quad -(28)$$

where ϕ is the gravitational potential:

$$\phi = -\frac{MG}{r} \quad -(29)$$

It follows that for a circular orbit:

6)

$$\frac{\partial \underline{v}}{\partial t} = \frac{\partial^2 \underline{r}}{\partial t^2} = 0 \quad -(30)$$

which is consistent with eq. (27).

Using the result:

$$\alpha = \frac{L^2}{m^2 m G} \quad -(31)$$

where L is the constant angular momentum of the system it follows that:

$$\begin{bmatrix} \omega^1{}_{01} & \omega^1{}_{02} \\ \omega^2{}_{01} & \omega^2{}_{02} \end{bmatrix} = \left(\frac{m}{L}\right)^3 (mG)^2 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(32)$$

Finally, the general force equation for all orbits is:

$$\boxed{\underline{F} = m\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -m\underline{\nabla}\phi} \quad -(33)$$