361(4): Analysis of the New Acceleration for the Plane Polar Coordinate System

The new acceleration is:

$$\underline{g}_1 = \left(\dot{r} \frac{\partial \dot{r}}{\partial r} + \dot{\theta} \frac{\partial \dot{r}}{\partial \theta} \right) \underline{e}_r + \left(r \dot{r} \frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2 \frac{\partial r}{\partial \theta} \right) \underline{e}_\theta \quad -(1)$$

In the plane polar coordinate system:

$$\underline{r} = X \underline{i} + Y \underline{j} = \underline{i}\, r\cos\theta + \underline{j}\, r\sin\theta \quad -(2)$$

so

$$\frac{\partial \underline{r}}{\partial r} = r\left(\underline{i}\cos\theta + \underline{j}\sin\theta\right) \quad -(3)$$

and

$$\frac{\partial \underline{r}}{\partial \theta} = \left(-\underline{i}\sin\theta + \underline{j}\cos\theta\right) r \quad -(4)$$

The unit vectors of the plane polar system are:

$$\underline{e}_r = \frac{\partial \underline{r}}{\partial r} \Big/ \left| \frac{\partial \underline{r}}{\partial r} \right| \quad -(5)$$

$$\underline{e}_\theta = \frac{\partial \underline{r}}{\partial \theta} \Big/ \left| \frac{\partial \underline{r}}{\partial \theta} \right| \quad -(6)$$

It follows that:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(7)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(8)$$

so

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \; ; \quad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \quad -(9)$$

) as used in the derivation of note 361(3).

Now note that:

$$\underline{e}_r = \frac{\partial \underline{r}}{\partial r} = \frac{\partial (r \underline{e}_r)}{\partial r} = \underline{e}_r + r \frac{\partial \underline{e}_r}{\partial r} \quad -(10)$$

It follows that $\frac{\partial \underline{e}_r}{\partial r} = \underline{0} \quad -(11)$

as used in the derivation of Note 361(3).

Secondly:

$$\underline{e}_\theta = \frac{1}{r}\frac{\partial \underline{r}}{\partial \theta} = \frac{1}{r}\frac{\partial (r \underline{e}_r)}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial r}{\partial \theta}\underline{e}_r + \frac{\partial \underline{e}_r}{\partial \theta} \quad -(12)$$

$$= \frac{1}{r}\frac{\partial r}{\partial \theta}\underline{e}_r + \underline{e}_\theta$$

It follows that in the plane polar coordinate system

$$\boxed{\frac{\partial r}{\partial \theta} = 0} \quad -(13)$$

by construction.

The reason for eq. (13) is to be found in the basic definitions:

$$X = r\cos\theta \quad -(14)$$

$$Y = r\sin\theta \quad -(15)$$

As r goes around in a circle, it does not change with θ in the plane polar coordinate system. Similarly there is no dependence of θ on r because θ is not a function of r, it is a function only of t, so:

$$\frac{\partial \theta}{\partial r} = 0 \quad -(16)$$

for the plane polar coordinate system.

Also, for a circular orbit:

$$r = \alpha = \text{constant} \quad -(17)$$

and

$$\frac{\partial r}{\partial t} = 0, \quad -(18)$$

so

$$\frac{\partial \theta}{\partial r} = \frac{\partial r}{\partial \theta}\left(\frac{\partial \theta}{\partial r}\right)^2 = 0. \quad -(19)$$

It follows that for the plane polar coordinate system:

$$\frac{\partial \dot{\theta}}{\partial r} = \frac{\partial \dot{\theta}}{\partial \theta}\frac{\partial \theta}{\partial r} = 0 \quad -(20)$$

and

$$\frac{\partial \dot{r}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{dr}{dt}\right) ; \quad \frac{\partial \dot{r}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{dr}{dt}\right) \quad -(21)$$

so the new acceleration becomes:

$$\boxed{\underline{g}_1 = \left(\dot{r}\frac{\partial}{\partial r}\left(\frac{dr}{dt}\right) + \dot{\theta}\frac{\partial}{\partial \theta}\left(\frac{dr}{dt}\right)\right)\underline{e}_r} \quad -(22)$$

It is radially directed and is a new type of orbital force.

If it is assumed that:

$$\frac{\partial}{\partial r}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{dr}{\partial r}\right) = 0 \quad -(23)$$

and

$$\frac{\partial}{\partial \theta}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{\partial r}{\partial \theta}\right) = 0 \quad -(24)$$

then

$$\boxed{\underline{g}_1 = \underline{0}} \quad -(25)$$

in the plane polar coordinate system, and we obtain:

$$\boxed{\begin{aligned} \underline{g} &= \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\,\underline{v} \qquad -(26) \\ &= \frac{\partial \underline{v}}{\partial t} - r\dot{\theta}^2 \underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \end{aligned}}$$

This result of dynamics unified with fluid dynamics shows that the Coriolis accelerations are due to the Lagrange derivative in the plane polar coordinate system.

In a more general coordinate system there emerges the possibility of many new orbital forces.