361(6): The Acceleration in Dynamics as a Cartan and Lagrange Derivative.

Consider the velocity $\underline{v}$ in plane polar coordinates:

$$\underline{v} = v_r \underline{e}_r + r\dot{\theta}\, \underline{e}_\theta \quad -(1)$$

$$= \dot{r}\, \underline{e}_r + v_\theta \underline{e}_\theta$$

$$= v_r \underline{e}_r + v_\theta \underline{e}_\theta$$

From ECE2 theory the acceleration is:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v} \quad -(2)$$

Note carefully that this is a general result valid for both dynamics and fluid dynamics. It is the result of ECE2 unified field theory, and is a new result of dynamics.

From previous notes:

$$\frac{D\underline{v}}{Dt} = \frac{\partial v_r}{\partial t}\underline{e}_r + \frac{\partial v_\theta}{\partial t}\underline{e}_\theta$$
$$+ \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} \right) \underline{e}_r \quad -(3)$$
$$+ \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} \right) \underline{e}_\theta$$

Now express eq. (3) as a covariant derivative of Cartan geometry:

$$\boxed{\underline{a} = \frac{D\underline{v}}{Dt} = \frac{Dv_r}{Dt}\underline{e}_r + \frac{Dv_\theta}{Dt}\underline{e}_\theta} \quad -(4)$$

2) Eq. (4) is a new and original definition of any acceleration.

Therefore:

$$\boxed{\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = \frac{Dv_r}{Dt}\,\underline{e}_r + \frac{Dv_\theta}{Dt}\,\underline{e}_\theta} \quad -(5)$$

The individual covariant derivatives are:

$$\frac{Dv_r}{Dt} = \frac{\partial v_r}{\partial t} + \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} \right)$$

$$= \frac{\partial v_r}{\partial t} + \dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} - r\dot{\theta}^2 \quad -(6)$$

and

$$\frac{Dv_\theta}{Dt} = \frac{\partial v_\theta}{\partial t} + \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} \right)$$

$$= \frac{\partial v_\theta}{\partial t} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2 \frac{\partial r}{\partial \theta} \right) \quad -(7)$$

Eqs. (6) and (7) can be expressed as the Cartan derivative:

$$\frac{D}{Dt}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} + \left(\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{v_\theta}{r} \\ \frac{v_\theta}{r} & 0 \end{bmatrix} \right) \begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad -(8)$$

where the quantity inside the brackets on the right hand side is the spin connection matrix.

) Therefore:

$$V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} = \dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} - r\dot{\theta}^2$$

$$= \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{1}{r}\frac{\partial V_r}{\partial \theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\frac{V_\theta}{r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} \quad -(9)$$

and

$$V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta V_r}{r} = r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2\frac{\partial r}{\partial \theta}$$

$$= \begin{bmatrix} 0 & 0 \\ \frac{\partial V_\theta}{\partial r} & \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{V_\theta}{r} & 0 \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} \quad -(10)$$

Note carefully that the new definition of acceleration, eq. (5), leads to new terms that are absent from the usual development:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(V_r \underline{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\,\underline{e}_\theta\right) \quad -(11)$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

which appear in any good textbook. The accelerations in eq. (11) were inferred by Coriolis in 1835. However eq. (5) leads to the new acceleration:

$$\boxed{\underline{a}_1 = \left(\dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta}\right)\underline{e}_r + \left(r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{\theta}^2\frac{\partial r}{\partial \theta}\right)\underline{e}_\theta}$$

$$-(12)$$

The accelerations (12) are unknown in classical dynamics, they are a fundamental new result of ECE2 unified field theory. They result from the velocity being considered as a velocity field as in fluid dynamics:

$$\underline{v} = \underline{v}(t, r(t), \theta(t)) \quad -(13)$$

rather than

$$\underline{v} = \underline{v}(t) \quad -(14)$$

in classical dynamics.

In the usual development of the plane polar coordinates (e.g. "Vector Analysis Problem Solver"):

$$\frac{\partial r}{\partial \theta} = 0 \quad -(15)$$

because r and θ are considered as the independent variables of the coordinate system (r, θ). Similarly, $\partial X / \partial Y = \partial X / \partial Z = \partial Y / \partial Z = 0$ in the Cartesian coordinate system.

However, if r and θ are functions of t, and $\underline{v}$ is a function of $r(t)$, $\theta(t)$ and t, eq. (15) is no longer necessarily true, and the coordinate system is generalized. For example, if the dynamics of the orbit is considered to be an idea of general coordinate system itself, as in general relativity, then eq. (15) is true for a circular orbit but it is not true in a conic section orbit or precessing ellipse.

In the usual plane polar coordinates eq (15) is
me for the coordinate system. This is slan in any good
textbook. Therefore:

$$\frac{\partial \dot{\theta}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{d\theta}{dt}\right) = \frac{\partial \omega}{\partial r} \quad -(16)$$

where the angular velocity of the coordinate system is

$$\omega = \dot{\theta} = \frac{d\theta}{dt} \quad -(17)$$

If this is constant then

$$\frac{\partial \dot{\theta}}{\partial r} = 0 \quad -(17)$$

and

$$\underline{a}_1 = \left(\dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta}\right)\underline{e}_r \quad -(18)$$

However if a coordinate system is used in which ω is not a constant, the acceleration:

$$\underline{a}_2 = r\dot{r}\frac{\partial \dot{\theta}}{\partial r}\,\underline{e}_\theta \quad -(19)$$

is not zero.

Now note that:

$$\frac{\partial \dot{r}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{dr}{dt}\right) = \frac{\partial v_r}{\partial \theta} \quad -(20)$$

and

$$\frac{\partial \dot{r}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{dr}{dt}\right) = \frac{\partial v_r}{\partial r} \quad -(21)$$

In the usual development of the acceleration
in plane polar coordinates:

$$v_r = \dot{r} = \frac{dr}{dt} = \frac{dr(t)}{dt} \quad -(22)$$

So v_r has no functional dependence on r or θ. In consequence:

$$\frac{\partial v_r}{\partial \theta} = \frac{\partial v_r}{\partial r} = 0 \quad -(23)$$

and

$$\underline{a_1 = 0} \quad -(24)$$

It follows that:

$$\underline{a} = \frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v}$$

$$= \frac{Dv_r}{Dt}\underline{e}_r + \frac{Dv_\theta}{Dt}\underline{e}_\theta$$

$$= (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(25)$$

Q.E.D.

However, eq. (25) is a very limited result that depends on:

$$\underline{v} = \underline{v}(t, r(t), \theta(t)) \rightarrow \underline{v}(t) \quad -(26)$$