1(3): The Convective or Lagrange Derivative in Plane Polar Coordinates: Spin Connection and Matrix Format

In plane polar coordinates the convective derivative is fixed as:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = \frac{\partial \underline{v}}{\partial t} + \left(v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_Z \frac{\partial}{\partial Z}\right)\left(v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_Z \underline{k}\right) \quad -(1)$$

$$= v_r \frac{\partial}{\partial r}(v_r \underline{e}_r) + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}(v_r \underline{e}_r) + v_Z \frac{\partial}{\partial Z}(v_r \underline{e}_r)$$

$$+ v_r \frac{\partial}{\partial r}(v_\theta \underline{e}_\theta) + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}(v_\theta \underline{e}_\theta) + v_Z \frac{\partial}{\partial Z}(v_\theta \underline{e}_\theta)$$

$$+ v_r \frac{\partial}{\partial r}(v_Z \underline{k}) + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}(v_Z \underline{k}) + v_Z \frac{\partial}{\partial Z}(v_Z \underline{k}) + \frac{\partial \underline{v}}{\partial t}$$

In general the derivatives of the quantities inside brackets must be worked out with the Leibnitz Theorem. In the plane polar system:

$$\frac{\partial \underline{k}}{\partial \theta} = \frac{\partial \underline{e}_r}{\partial r} = \frac{\partial \underline{e}_r}{\partial Z} = \frac{\partial \underline{e}_\theta}{\partial Z} = \frac{\partial \underline{e}_r}{\partial Z} = \frac{\partial \underline{k}}{\partial Z} = 0 \quad -(2)$$

and

$$\partial r / \partial \theta = 0. \quad -(3)$$

By construction:

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \; ; \; \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \quad -(4)$$

This means that:

2)

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial Z} \right) \underline{e}_r + \frac{v_\theta v_r}{r}\frac{\partial \underline{e}_r}{\partial \theta}$$

$$+ \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial Z} \right) \underline{e}_\theta + \frac{v_\theta^2}{r}\frac{\partial \underline{e}_\theta}{\partial r}$$

$$+ \left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial Z} \right) \underline{k} \quad -(5)$$

$$= \frac{\partial \mathbf{v}}{\partial t} + \left(\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial Z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} & \frac{\partial v_\theta}{\partial Z} \\ \frac{\partial v_z}{\partial r} & \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial Z} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{v_\theta}{r} & 0 \\ \frac{v_\theta}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix}$$

in cylindrical polar coordinates.

In the plane polar coordinates:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \left(\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{v_\theta}{r} \\ \frac{v_\theta}{r} & 0 \end{bmatrix} \right) \begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad -(6)$$

The spin connection matrix of the plane polar system of coordinates is therefore:

$$\omega^a{}_{b\,i} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{v_\theta}{r} \\ \frac{v_\theta}{r} & 0 \end{bmatrix} \quad -(7)$$

) and the convective derivative is the Cartan derivative:

$$\frac{Dv^a}{Dt} = \frac{\partial v^a}{\partial t} + \omega^a{}_{ob} v^b \quad -(8)$$

Eq. (8) is intended to mean:

$$\frac{D}{Dt}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} + \left(\begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{1}{r}\frac{\partial V_r}{\partial \theta} \\ \frac{\partial V_\theta}{\partial r} & \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{V_\theta}{r} \\ \frac{V_\theta}{r} & 0 \end{bmatrix} \right) \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} \quad -(9)$$

Denoting:

$$v^1 = V_r \; ; \; v^2 = V_\theta \quad -(10)$$

then:

$$\frac{Dv^1}{Dt} = \frac{\partial v^1}{\partial t} + \left(\left(\frac{\partial v^1}{\partial r} \right) v^1 + \frac{1}{r}\left(\frac{\partial v^1}{\partial \theta} \right) v^2 - \frac{v^{22}}{r} \right)$$

$$= \frac{\partial v^1}{\partial t} + \left(\left(\frac{\partial v^1}{\partial r} \right) v^1 + \left(\frac{\partial v^1}{\partial \theta} - \frac{v^2}{r} \right) v^2 \right) \quad -(11)$$

$$= \frac{\partial v^1}{\partial t} + \omega^1{}_{01} v^1 + \omega^1{}_{02} v^2$$

So

$$\omega^1{}_{01} = \frac{\partial v^1}{\partial r} = \frac{\partial V_r}{\partial r} \quad -(12)$$

$$\omega^1{}_{02} = \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} \quad -(13)$$

Similarly:

$$\omega^2{}_{01} = \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \quad -(14)$$

$$\omega^2{}_{02} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \quad -(15)$$

4) Eqs. (14) and (15) are worked out from:

$$\frac{Dv^2}{Dt} = \frac{\partial v^2}{\partial t} + \omega^2{}_{01} v^1 + \omega^2{}_{02} v^2 \quad -(16)$$

i.e

$$\frac{DV_\theta}{Dt} = \frac{\partial V_\theta}{\partial t} + \omega^2{}_{01} V_r + \omega^2{}_{02} V_\theta \quad -(17)$$

$$= \frac{\partial V_\theta}{\partial t} + \left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\right) V_r + \left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}\right) V_\theta$$

So eqs. (14) and (15) follow, Q.E.D.

Now use:

$$V_r = \dot{r}, \quad V_\theta = r\dot{\theta} \quad -(18)$$

to find that:

$$\begin{bmatrix} 0 & -V_\theta/r \\ V_\theta/r & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \quad -(19)$$

i.e. the spin connection that defines the angular velocity:

$$\dot{\theta} = \frac{d\theta}{dt}. \quad -(20)$$

This means that the plane polar coordinate system is a moving frame, specifically a rotating frame.

Therefore the spin connection components are:

$$\omega^1{}_{01} = \partial\dot{r}/\partial r \quad -(21)$$

$$\omega^1{}_{02} = \frac{1}{r}\frac{\partial\dot{r}}{\partial\theta} - \dot{\theta} \quad -(22)$$

$$\omega^2{}_{01} = \frac{\partial(r\dot{\theta})}{\partial r} + \dot{\theta} \quad -(23)$$

$$\omega^2 \quad = \frac{1}{\cdot}\frac{\partial(r\dot{\theta})}{\partial\theta} \quad -(24)$$