359(8) : Elliptical Orbit.

From note 359(7):

$$X = \frac{mG}{v^3}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right) \quad -(1)$$

$$Y = \frac{mG}{v^3}\left(r\dot{\theta}\sin\theta - \dot{r}\cos\theta\right) \quad -(2)$$

where

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \left(\frac{2}{r} - \frac{1}{a}\right)mG \quad -(3)$$

Here a is the semi major axis:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(4)$$

So:

$$\frac{1}{(X^2+Y^2)^{1/2}} = \frac{2}{r} - \frac{1}{a} \quad -(5)$$

For a circular orbit:

$$v^2 = \frac{mG}{r} \quad -(6)$$

so

$$X^2 + Y^2 = r^2 \quad -(7)$$

Q.E.D.

However, for an elliptical orbit:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad -(8)$$

where

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} \quad -(9)$$

2) For an ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(10)$$

so

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \quad -(11)$$

From a Lagrangian analysis:

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \quad -(12)$$

where

$$\mu = \frac{mM}{m+M} \sim m \quad -(13)$$

and L is the constant angular momentum.

We also have:

$$L^2 = m^2 M G \alpha. \quad -(14)$$

It follows that:

$$X = \alpha^{1/2} \frac{\left(\frac{\epsilon}{\alpha} \sin^2\theta + \frac{1}{r}\cos\theta\right)}{\left(\frac{2}{r} - \frac{1}{a}\right)^{3/2}} \quad -(15)$$

$$Y = \alpha^{1/2} \frac{\left(\frac{1}{r}\sin\theta - \frac{\epsilon \sin\theta \cos\theta}{\alpha}\right)}{\left(\frac{2}{r} - \frac{1}{a}\right)^{3/2}} \quad -(16)$$

Using eq. (10), X and Y can be plotted against θ, or against r.

3) For a circle:

$$\epsilon = 0, \quad \frac{2}{r} - \frac{1}{a} = \frac{1}{r}, \quad r = \alpha \quad -(17)$$

So:

$$X = r\cos\theta \quad -(18)$$
$$Y = r\sin\theta \quad -(19)$$

and

$$X^2 + Y^2 = r^2 \quad -(20)$$

Q.E.D.

Therefore eqns. (15) and (16) define X and Y for a planar orbit when that orbit is an ellipse.

The method holds for any planar orbit QED