359(3): Details of the Three Dimensional Calculation

Consider the velocity fields:

$$\underline{v}_{F1} = \frac{A}{(X^2+Y^2)^{3/2}} \left(-Y\underline{i} + X\underline{j} \right) \quad -(1)$$

$$\underline{v}_{F2} = \frac{A}{(X^2+Z^2)^{3/2}} \left(-Z\underline{i} + X\underline{k} \right) \quad -(2)$$

$$\underline{v}_{F3} = \frac{A}{(Y^2+Z^2)^{3/2}} \left(-Z\underline{j} + Y\underline{k} \right) \quad -(3)$$

In three dimensions:

$$\underline{\nabla} = \underline{i}\frac{\partial}{\partial X} + \underline{j}\frac{\partial}{\partial Y} + \underline{k}\frac{\partial}{\partial Z} \quad -(4)$$

It follows that:

$$\underline{v}_{F1}\cdot\underline{\nabla} = \frac{A}{(X^2+Y^2)^{3/2}} \left(-Y\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y} \right) \quad -(5)$$

$$\underline{v}_{F2}\cdot\underline{\nabla} = \frac{A}{(X^2+Z^2)^{3/2}} \left(-Z\frac{\partial}{\partial X} + X\frac{\partial}{\partial Z} \right) \quad -(6)$$

$$\underline{v}_{F3}\cdot\underline{\nabla} = \frac{A}{(Y^2+Z^2)^{3/2}} \left(-Z\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z} \right) \quad -(7)$$

Now define:

$$\underline{g}_1 = \left(\underline{v}_{F1}\cdot\underline{\nabla} \right) \underline{v}_{F1} \quad -(8)$$

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$$\underline{g}_2 = (\underline{v}_{F2} \cdot \underline{\nabla}) \underline{v}_{F2} \quad -(9)$$

$$\underline{g}_3 = (\underline{v}_{F3} \cdot \underline{\nabla}) \underline{v}_{F3} \quad -(10)$$

As in Note 359(1):

$$\underline{g}_1 = -\frac{mG}{(X^2+Y^2)^{3/2}} (X\underline{i} + Y\underline{j}) \quad -(11)$$

using

$$A^2 = mG. \quad -(12)$$

Now consider:

$$(\underline{v}_{F2} \cdot \underline{\nabla}) \underline{v}_{F2} = A^2 f_2 \left(-Z\frac{\partial}{\partial X} + X\frac{\partial}{\partial Z}\right)\left(f_2 \left(-Z\underline{i} + X\underline{k}\right)\right).$$

$$= A^2 f_2 \left(-Z\frac{\partial}{\partial X} + X\frac{\partial}{\partial Z}\right)\left(-Z\underline{i} + X\underline{k}\right)$$

$$+ A^2 f_2 \left(-Z\underline{i} + X\underline{k}\right)\left(-Z\frac{\partial}{\partial X} + X\frac{\partial}{\partial Z}\right) f_2 \quad -(13)$$

Note that:

$$-Z\frac{\partial f_2}{\partial X} + X\frac{\partial f_2}{\partial Z} = 0 \quad -(14)$$

because:

$$f_2 = (X^2+Z^2)^{-3/2} \quad -(15)$$

so

$$\frac{\partial f_2}{\partial X} = -\frac{2X(X^2+Z^2)^{1/2}}{(X^2+Z^2)^3} \quad -(16)$$

and:
$$\frac{\partial f_2}{\partial z} = -\frac{2z\left(x^2+z^2\right)^{1/2}}{\left(x^2+z^2\right)^3} \quad -(17)$$

Q.E.D

So:

$$\underline{g}_2 = \left(\underline{v}_{F2}\cdot\underline{\nabla}\right)\underline{v}_{F2}$$
$$= \frac{A^2}{\left(x^2+z^2\right)^{3/2}}\left(-z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z}\right)\left(-z\underline{i}+x\underline{k}\right)$$
$$= -\frac{mG}{\left(x^2+z^2\right)^{3/2}}\left(x\underline{i}+z\underline{k}\right) \quad -(18).$$

Finally consider:

$$\left(\underline{v}_{F3}\cdot\underline{\nabla}\right)\underline{v}_{F3} = A^2 f_3\left(-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)\left(f_3\left(-z\underline{j}+y\underline{k}\right)\right)$$
$$= A^2 f_3\left(-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)\left(-z\underline{j}+y\underline{k}\right)$$
$$+A^2 f_3\left(-z\underline{j}+y\underline{k}\right)\left(-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)f_3$$
$$= -\frac{mG}{\left(y^2+z^2\right)^{3/2}}\left(y\underline{j}+z\underline{k}\right) \quad -(19)$$

Therefore:

Therefore:

$$\underline{g}_1 + \underline{g}_2 + \underline{g}_3$$

$$= -mG\left(\frac{X\underline{i} + Y\underline{j}}{(X^2+Y^2)^{3/2}} + \frac{X\underline{i} + Z\underline{k}}{(X^2+Z^2)^{3/2}} + \frac{Y\underline{j} + Z\underline{k}}{(Y^2+Z^2)^{3/2}}\right)$$

$$= (\underline{v}_{F1}\cdot\underline{\nabla})\underline{v}_{F1} + (\underline{v}_{F2}\cdot\underline{\nabla})\underline{v}_{F2} + (\underline{v}_{F3}\cdot\underline{\nabla})\underline{v}_{F3} \quad -(20)$$

The Newtonian gravitational field is defined as:

$$\underline{g} = \frac{1}{2(X^2+Y^2+Z^2)^{3/2}}\left((X^2+Y^2)^{3/2}\underline{g}_1 + (X^2+Z^2)^{3/2}\underline{g}_2 + (Y^2+Z^2)^{3/2}\underline{g}_3\right)$$

$$= -\frac{mG}{r^3}\underline{r} \quad -(21)$$

where $\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k}$. $-(22)$

i.e. $\underline{g} = -\frac{mG}{r^2}\underline{e}_r$, $-(23)$

where $\underline{r} = r\,\underline{e}_r$ $-(24)$