351(5): Derivation of the Stokes Derivative from Cartan Geometry

In fluid dynamics the Stokes derivative is:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \underline{\nabla}\rho \cdot \underline{v} \quad -(1)$$

$$= \frac{\partial\rho}{\partial X}\frac{dX}{dt} + \frac{\partial\rho}{\partial Y}\frac{dY}{dt} + \frac{\partial\rho}{\partial Z}\frac{dZ}{dt} + \frac{\partial\rho}{\partial t}$$

where the fluid density is:

$$\rho = \rho(X, Y, Z, t) \quad -(2)$$

and

$$X = X(t),\ Y = Y(t),\ Z = Z(t) \quad -(3)$$

Therefore the frame (X, Y, Z) is time dependent, and is a moving frame. ECE2 and Cartan geometry also consider moving frames.

The Stokes derivative is therefore described in general by the covariant derivative of Cartan geometry:

$$\frac{DV^a}{dx^\mu} = \frac{\partial V^a}{\partial x^\mu} + \omega^a_{\ \mu b} V^b \quad -(4)$$

where $\omega^a_{\ \mu b}$ is the spin connection. So the fundamentals of fluid dynamics are described by a Cartan spin connection, because a translating frame is being considered.

Define the four vector:

$$V^\mu = (c\rho, \underline{v}\rho) \quad -(5)$$

) and consider the indices:

$$\mu = 0, \quad a = 0. \quad -(6)$$

It follows that in Cartan geometry:

$$\frac{D\rho}{\partial t} = \frac{\partial \rho}{\partial t} + \omega^0{}_{01}\rho \frac{dX}{dt} + \omega^0{}_{02}\rho \frac{dY}{dt} + \omega^0{}_{03}\rho \frac{dZ}{dt} \quad -(7)$$

This is the Stokes derivative provided that:

$$\omega^0{}_{01}\rho = \frac{\partial \rho}{\partial X}, \quad \omega^0{}_{02}\rho = \frac{\partial \rho}{\partial Y}, \quad \omega^0{}_{03}\rho = \frac{\partial \rho}{\partial Z} \quad -(8)$$

i.e.

$$\boxed{\underline{\nabla}\rho = \underline{\omega}\rho} \quad -(9)$$

where

$$\underline{\omega} = \omega^0{}_{01}\,\underline{i} + \omega^0{}_{02}\,\underline{j} + \omega^0{}_{03}\,\underline{k} \quad -(10)$$

and

$$\underline{\nabla} = \underline{i}\frac{\partial}{\partial X} + \underline{j}\frac{\partial}{\partial Y} + \underline{k}\frac{\partial}{\partial Z} \quad -(11)$$

The Cartan derivative is related to the Stokes derivative by:

$$\boxed{\frac{D\rho}{Dt}(\text{Stokes}) = \frac{D\rho}{\partial t}(\text{Cartan})} \quad -(12)$$

Similarly, the convective derivative of fluid dynamics

) is: $$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{\nabla} \cdot \underline{v})\,\underline{v} \quad -(13)$$

nd is also an example of Cartan geometry. This will be
developed in detail in the next note.

Therefore the Stokes derivative is:

$$\boxed{\frac{D\rho}{\partial t} = \frac{\partial\rho}{\partial t} + \underline{v} \cdot \underline{\omega}\,\rho} \quad -(14)$$

If considering the convective derivative, the velocity
flow field: $$\underline{v} = \underline{v}(X, Y, Z, t) \quad -(15)$$

is considered, rather than the density ρ.