351(3): Comparison of Continuity Equations

The conventional continuity equation of fluid dynamics is:

$$\frac{\partial \rho_m}{\partial t} + \underline{\nabla} \cdot \underline{J}_m = 0 \quad -(1)$$

where ρ_m is the mass density and $\underline{J_m}$ the current of mass density:

$$\underline{J_m} = \rho_m \underline{v} \quad -(2)$$

where

$$\underline{v} = \underline{v}(x, t) \quad -(3)$$

is the velocity field. From eqs. (1) and (2):

$$\frac{\partial \rho_m}{\partial t} + \rho_m \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla} \rho_m = 0 \quad -(4)$$

Kambe transforms eq. (1) into:

$$\frac{\partial q}{\partial t} + \underline{\nabla} \cdot \underline{J}_F = 0 \quad -(5)$$

where

$$q = \underline{\nabla} \cdot ((\underline{v} \cdot \underline{\nabla}) \underline{v}) \quad -(6)$$

and

$$\underline{J}_F = \frac{\partial^2 \underline{v}}{\partial t^2} + \underline{\nabla} \frac{\partial h}{\partial t} + a_0^2 \underline{\nabla} \times (\underline{\nabla} \times \underline{v}) \quad -(7)$$

where h is the enthalpy per kilogram in units of joules kg^{-1} and a_0 is the assumed constant speed of sound. The continuity equation (5) is equivalent to:

$$\frac{\partial h}{\partial t} + \underline{v} \cdot \underline{\nabla} h + a_0^2 \underline{\nabla} \cdot \underline{v} = 0 \quad -(8)$$

The Kanse transformation is accomplished with the [illegible]sions:

$$\frac{\partial \rho}{\partial t} = \frac{\rho}{a_0^2} \frac{\partial h}{\partial t} \quad -(9)$$

and

$$\underline{\nabla} \rho = \frac{\rho}{a_0^2} \underline{\nabla} h \quad -(10)$$

so eq. (4) becomes:

$$\frac{\rho}{a_0^2}\left(\frac{\partial h}{\partial t} + \underline{v} \cdot \underline{\nabla} h + a_0^2 \underline{\nabla} \cdot \underline{v}\right) = 0 \quad -(11)$$

which is eq. (8), QED.

The continuity equation of electrodynamics is

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0 \quad -(12)$$

where ρ is the charge density and $\underline{J}$ the current density. From Note 351(2):

$$\rho = \frac{\epsilon_0 \rho_m}{\rho} q \quad -(13)$$

From eq. (5) it follows that in the first approximation:

$$\underline{J} = \epsilon_0 \underline{\rho_m} \, \underline{J}_F \quad -(14)$$

Eq. (14) is the correct version of eq. (18) of note [illegible]1(2). More accurately:

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0 \rho_m}{\rho} q\right) + \underline{\nabla} \cdot \underline{J} = 0 \quad -(15)$$

3) Therefore the electromagnetic charge density is:

$$\boxed{\rho^2 = \epsilon_0 \rho_m q_v = \epsilon_0 \rho_m \underline{\nabla} \cdot \left(\left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} \right)} \quad - (16)$$

and charge becomes the property of a fluid field. The electromagnetic current density is defined by:

$$\underline{\nabla} \cdot \underline{J} = - \frac{\partial \rho}{\partial t} \quad - (17)$$

where ρ is defined by eq. (16). So $\underline{J}$ also becomes the property of a fluid field.

Summary of Fluid Electrodynamics to Date.

1) $$\frac{D\underline{\overline{W}}}{Dt} = \frac{\partial \underline{\overline{W}}}{\partial t} + \frac{1}{\rho_m} \left(\underline{\overline{W}} \cdot \underline{\nabla} \right) \underline{\overline{W}} = \underline{\nabla} \phi_W \quad - (18)$$

which is the electrodynamic Euler equation.

2) The continuity equation:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0 \quad - (19)$$

where

$$\rho^2 = \epsilon_0 \rho_m \underline{\nabla} \cdot \left(\left(\underline{v} \cdot \underline{\nabla} \right) v \right) \quad - (20)$$

3) The vortex equation

$$\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \left(\underline{B} \times \underline{v} \right) = \frac{1}{R} \nabla^2 \underline{B} \quad - (21)$$

4) These are completely new equations of classical electrodynamics and merge the subject with fluid dynamics.

Eq. (21) can be developed with:

$$\underline{B} = \underline{\nabla} \times \underline{\overline{W}} \quad -(22)$$

where

$$\underline{\overline{W}} = \frac{\rho_m}{\rho} \underline{v} \quad -(23)$$

The Euler equation is:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = \underline{\nabla} h \quad -(24)$$

in which both sides have the correct units of $m s^{-2}$, or acceleration. The most accurate translation of eq. (24) into fluid electrodynamics is made with

$$\underline{v} = \frac{\rho}{\rho_m} \underline{\overline{W}} \quad -(25)$$

and

$$h = \frac{\rho}{\rho_m} \phi_{\overline{W}} \quad -(26)$$

where

$$\overline{W}^\mu = \left(\frac{\phi_W}{c}, \underline{\overline{W}} \right) \quad -(27)$$

So the most accurate form of eq. (4) is:

5)

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_m}\overline{W}\right)+\frac{\rho}{\rho_m}\left(\overline{W}\cdot\overline{\nabla}\right)\frac{\rho}{\rho_m}\overline{W} = \underline{\nabla}\left(\frac{\rho}{\rho_m}\phi_W\right)} \quad -(28)$$

Eq. (28) is roughly approximated by eq. (18) if ρ/ρ_m is approximately independent of space and time, or if ρ/ρ_m varies slowly in space and time. Then ρ/ρ_m can be regarded as a constant and

$$\boxed{\frac{\partial\overline{W}}{\partial t}+\frac{\rho}{\rho_m}\left(\overline{W}\cdot\overline{\nabla}\right)\overline{W} = \overline{\nabla}\phi_W} \quad -(29)$$

Similarly, the vorticity equation is constructed from

$$\underline{w} = \underline{\nabla}\times\underline{v} \quad -(30)$$

and

$$\frac{\partial\underline{w}}{\partial t}+\underline{\nabla}\times\left(\underline{w}\times\underline{v}\right) = \frac{1}{R}\nabla^2\underline{w} \quad -(31)$$

where R is the Reynold number. The magnetic flux density is

$$\underline{B} = \underline{\nabla}\times\overline{W} \quad -(32)$$

where

$$\overline{W} = \frac{\rho_m}{\rho}\underline{v} \quad -(33)$$

so

$$\underline{B} = \underline{\nabla}\times\left(\frac{\rho_m}{\rho}\underline{v}\right) \quad -(34)$$

and

$$\underline{W} = \underline{\nabla} \times \left(\frac{\rho}{\rho_m} \underline{W} \right) \quad -(35)$$

If ρ / ρ_m is slowly varying over space and time:

$$\underline{W} = \frac{\rho}{\rho_m} \underline{\nabla} \times \underline{W} = \frac{\rho}{\rho_m} \underline{B} \quad -(36)$$

and ρ / ρ_m can be regarded as a constant. In this case eq. (31) becomes eq. (21) Q.E.D.

Note that the factor ρ / ρ_m of eq. (29) was accidentally omitted from eq. (34) of Note 351(2).

Finally:

$$\overline{W} = W^{(0)} \underline{\omega} \quad -(37)$$

where $\underline{\omega}$ is the spin connection, so the equations in $\overline{W}$ are also equations in the spin connection, and therefore equations of spacetime itself.