351(6): Derivation of the Convective Derivative of Fluid Dynamics from Cartan Geometry.

The convective derivative of velocity for example is

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} \quad -(1)$$

where

$$\underline{v} = \underline{v}(X, Y, Z, t) \quad -(2)$$

X, Y and Z are functions of the velocity flow. In general, X, Y and Z are functions of time and distance.

The covariant derivative of Cartan geometry is:

$$\frac{DV^a}{\partial x^\mu} = \frac{\partial V^a}{\partial x^\mu} + \omega^a{}_{\mu b} V^b \quad -(3)$$

where

$$x^\mu = (ct, X, Y, Z) \quad -(4)$$

Consider the $\mu = 0$ component in eq. (3), and defining:

$$V^a = \left(\frac{E}{m}, c\underline{v}\right) \quad -(5)$$

it is found that:

$$\frac{DV^1}{\partial x^0} = \frac{\partial V^1}{\partial x^0} + \omega^1{}_{01} V^1 + \omega^1{}_{02} V^2 + \omega^1{}_{03} V^3$$

$$\frac{DV^2}{\partial x^0} = \frac{\partial V^2}{\partial x^0} + \omega^2{}_{01} V^1 + \omega^2{}_{02} V^2 + \omega^2{}_{03} V^3$$

$$\frac{DV^3}{\partial x^0} = \frac{\partial V^3}{\partial x^0} + \omega^3{}_{01} V^1 + \omega^3{}_{02} V^2 + \omega^3{}_{03} V^3$$

$$-(6)$$

) The convective derivative is defined by:

$$\omega^1{}_{02} = \omega^1{}_{03} = \omega^2{}_{01} = \omega^2{}_{03} = \omega^3{}_{01} = \omega^3{}_{02} = 0 \quad -(7)$$

So

$$\frac{D\bar{V}_x}{\partial t} = \frac{\partial \bar{V}_x}{\partial t} + \omega^1{}_{01} \bar{V}_x \quad -(8)$$

$$\frac{D\bar{V}_y}{\partial t} = \frac{\partial \bar{V}_y}{\partial t} + \omega^2{}_{02} \bar{V}_y \quad -(9)$$

$$\frac{D\bar{V}_z}{\partial t} = \frac{\partial \bar{V}_z}{\partial t} + \omega^3{}_{03} \bar{V}_z \quad -(10)$$

Therefore:

$$\omega^1{}_{01} = \frac{\partial v_x}{\partial X} = (\underline{v} \cdot \underline{\nabla})_x \quad -(11)$$

$$\omega^2{}_{02} = \frac{\partial v_y}{\partial Y} = (\underline{v} \cdot \underline{\nabla})_y \quad -(12)$$

$$\omega^3{}_{03} = \frac{\partial v_z}{\partial Z} = (\underline{v} \cdot \underline{\nabla})_z \quad -(13)$$

and

$$\boxed{\omega^0 = \underline{v} \cdot \underline{\nabla}} \quad -(14)$$

[...]ere:

$$\omega^0 = \omega^1{}_{01} + \omega^2{}_{02} + \omega^3{}_{03}$$

[...]d so the convective derivative is: $-(15)$

$$\boxed{\begin{aligned}\frac{D\underline{v}}{Dt} &= \frac{\partial \underline{v}}{\partial t} + \omega^0 \underline{v} \\ &= \left(\frac{\partial}{\partial t} + \omega^0\right)\underline{v}\end{aligned}} \quad -(16)$$

and is defined by a scalar spin connection.

The Stokes derivative is:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \underline{v}\cdot\underline{\omega}\,\rho \quad -(17)$$

where

$$\underline{\omega}\,\rho = \underline{\nabla}\rho \quad -(18)$$

and is defined by a vector spin connection. Therefore it is possible to define:

$$\boxed{\begin{aligned}\omega^\mu &= \left(\frac{\omega^0}{c}, \underline{\omega}\right) = \partial^\mu \\ &= \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right)\end{aligned}} \quad -(19)$$

For incompressible fluids without viscosity:

$$\omega^0 = \underline{\nabla}\cdot\underline{v} = 0. \quad -(20)$$

The spin connection ω^0 is therefore responsible for shear flow and the Reynolds number, and therefore the transition to turbulence.

+) The meaning of the spin connection in fluid dynamics is that the fluid is a continuum in which X, t and Z, are dependent on time t and on distance. Therefore the frame of reference can be considered as moving.

We have:

$$\boxed{\omega^0 = \underline{v} \cdot \underline{\nabla} = \underline{v} \cdot \underline{\omega}} \quad -(21)$$

Therefore in all the equations of hydrodynamics, $\partial / \partial t$ can be replaced by ω^0 and $\underline{\nabla}$ by $\underline{\omega}$.

The vorticity is therefore:

$$\underline{w} = \underline{\nabla} \times \underline{v} = \underline{\omega} \times \underline{v}. \quad -(22)$$

in which the curl operator is:

$$\underline{\nabla} \times = \underline{\omega} \times \quad -(23)$$

The equation for the Reynolds number is:

$$\boxed{\begin{aligned} \frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{w} \times \underline{v}) &= \frac{1}{R} \nabla^2 \underline{w} \\ = \omega^0 \underline{w} + \underline{\omega} \times (\underline{w} \times \underline{v}) &= \frac{1}{R} \omega^2 \underline{w} \end{aligned}}$$

$$-(24)$$