351(8): Incorporating the Vorticity in the Kambe Field Equations

The vorticity is defined in the Kambe field equations by:

$$\underline{H} = \underline{w} = \underline{\nabla} \times \underline{v} \quad -(1)$$

So:

$$\underline{\nabla} \cdot \underline{H} = \underline{\nabla} \cdot \underline{w} = 0 \quad -(2)$$

The vorticity equation used by Kambe comes from conservation of angular momentum of flow:

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{w} \times \underline{v}) = 0 \quad -(3)$$

The homogeneous field equation of Kambe is:

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{w}}{\partial t} = \underline{0} \quad -(4)$$

where

$$\underline{E} = (\underline{v} \cdot \underline{\nabla}) \underline{v} \quad -(5)$$

Therefore the convective derivative is defined by:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = \frac{\partial \underline{v}}{\partial t} + \underline{E} \quad -(6)$$

So from Note 351(7), $\underline{E}$ is related to the Jacobian or spin connection:

$$\omega^a{}_{b} = \frac{\partial v^a}{\partial r^b} \quad -(7)$$

From eqs. (3) and (4):

$$\boxed{\underline{E} = \underline{v} \times \underline{w} = (\underline{v} \cdot \underline{\nabla}) \underline{v}} \quad -(8)$$

For a Beltrami flow:

$$\underline{\nabla} \times \underline{v} = k\underline{v} \quad - (9)$$

so

$$(\underline{v} \cdot \underline{\nabla})\underline{v} = \underline{0} \quad - (10)$$

and for a Beltrami flow:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} \quad - (11)$$

and the spin connection or Jacobian is zero:

$$\omega^a{}_{ob} = \frac{\partial v^a}{\partial r^b} = 0. \quad - (12)$$

For the general flow from eqs. (3) and (8) the convective derivative is:

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \times \underline{w} \quad - (13)$$

and eq. (8) must be solved for $\underline{v}$:

$$\boxed{\underline{v} \times (\underline{\nabla} \times \underline{v}) = (\underline{v} \cdot \underline{\nabla})\underline{v}} \quad - (14)$$

The Reynolds number enters into the analysis by generalizing eq. (3) to:

$$\frac{\partial \underline{w}}{\partial t} + \underline{\nabla} \times (\underline{w} \times \underline{v}) = \frac{1}{R}\nabla^2 \underline{w} \quad - (15)$$

so:

$$\underline{\nabla} \times \underline{E} = \frac{1}{R}\nabla^2 \underline{w} - \underline{\nabla} \times (\underline{w} \times \underline{v}) \quad - (16)$$

Now use:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{W}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{W}) - \nabla^2 \underline{W} \quad -(17)$$

and

$$\underline{\nabla} \cdot \underline{W} = 0 \quad -(18)$$

So

$$\nabla^2 \underline{W} = -\underline{\nabla} \times (\underline{\nabla} \times \underline{W}) \quad -(19)$$

It follows from eq. (16) and (19) that:

$$\boxed{\begin{aligned} \underline{E} &= \underline{v} \times \underline{W} - \frac{1}{R} \underline{\nabla} \times \underline{W} \\ &= (\underline{v} \cdot \underline{\nabla}) \underline{v} \end{aligned}} \quad -(20)$$

Therefore transition to turbulence is governed in general by eq. (20):

$$\boxed{\underline{\nabla} \times \underline{W} = R\left((\underline{v} \cdot \underline{\nabla})\underline{v} - \underline{v} \times \underline{W}\right)} \quad -(21)$$

Turbulence in Beltrami flows is defined by:

$$\boxed{\underline{\nabla} \times \underline{W} = R(\underline{v} \cdot \underline{\nabla})\underline{v}} \quad -(22)$$

Here:

$$\begin{aligned} \underline{\nabla} \times \underline{W} &= \underline{\nabla} \times (\underline{\nabla} \times \underline{v}) \\ &= \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) - \nabla^2 \underline{v} \end{aligned} \quad -(23)$$

Fm eqs. (22) and (23):

$$\boxed{\nabla^2 \underline{v} = R(\underline{v} \cdot \underline{\nabla})\underline{v} - \underline{\nabla}(\underline{\nabla} \cdot \underline{v})} \quad -(24)$$