349(3): Condition for Vanishing Magnetic Charge / Current Density

It was shown in note 349(3) that this condition is:

$$r^{(0)}\underline{\omega}\cdot\underline{\nabla}\times\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\times\underline{q} - \underline{\nabla}\cdot\underline{q}\times\underline{\omega} \quad (1)$$

Now consider the Cartan identity in vector notation (PECE chapter 3):

$$\underline{\nabla}\cdot\underline{\omega}^a{}_b\times\underline{q}^b = \underline{q}^b\cdot\underline{\nabla}\times\underline{\omega}^a{}_c - \underline{\omega}^a{}_b\cdot\underline{\nabla}\cdot\underline{q}^b \quad -(2)$$

The procedure of removing indices reduces this to the well known vector identity:

$$\underline{\nabla}\cdot\underline{\omega}\times\underline{q} = \underline{q}\cdot\underline{\nabla}\times\underline{\omega} - \underline{\omega}\cdot\underline{\nabla}\times\underline{q} \quad -(3)$$

From eqs. (1) and (2):

$$\boxed{\underline{\nabla}\cdot\underline{q}\times\underline{\omega} = r^{(0)}\underline{\omega}\cdot\underline{\nabla}\times\underline{\omega}} \quad -(4).$$

Under this condition there is no magnetic charge current density in ECE2 theory.

In ECE2:

$$\underline{A} = A^{(0)}\underline{q} \quad -(5)$$

and

$$\underline{W} = W^{(0)}\underline{\omega} \quad -(6)$$

The units of $\underline{\omega}$ are m^{-1}, q is unitless, $\underline{A}$ is tesla metres, and $\underline{W}$ is m^2 tesla. So:

$$\frac{A^{(0)}}{W^{(0)}} = \frac{1}{m^2} = \frac{1}{r^{(0)2}} \quad -(7)$$

and

$$\frac{|\underline{A}|}{|\underline{W}|} = \frac{1}{r^{(0)}} \quad -(8)$$

2) It follows that eq. (4) is equivalent to:

$$\boxed{\underline{\nabla} \cdot \underline{A} \times \underline{W} = \frac{1}{r^{(0)}} \underline{W} \cdot \underline{\nabla} \times \underline{W}} \quad -(9)$$

In ECE2:

$$\underline{B} = \underline{\nabla} \times \underline{W} \quad -(10)$$

so

$$\underline{\nabla} \cdot \underline{A} \times \underline{W} = \frac{1}{r^{(0)}} \underline{W} \cdot \underline{B} \quad -(11)$$

Eq. (9) implies that

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(12)$$

because Eq. (9) implies no magnetic monopole. A solution of Eq. (12) is:

$$\underline{\nabla} \times \underline{B} = k \underline{B} \quad -(13)$$

where k has the units of inverse metres. This is because:

$$\underline{B} = \frac{1}{k} \underline{\nabla} \times \underline{B} \quad -(14)$$

and

$$\underline{\nabla} \cdot \underline{B} = \frac{1}{k} \underline{\nabla} \cdot \underline{\nabla} \times \underline{B} = 0 \quad -(15)$$

QED. Eq. (13) is the Beltrami solution.

It implies that

$$\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{W}) \quad -(16)$$

$$= k \underline{\nabla} \times \underline{W}$$

A possible solution of Eq. (16) is the Trkalian field (chapter 3 of PECE):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\overline{W}}) = k \underline{\nabla} \times \underline{\overline{W}} = k^2 \underline{\overline{W}} \quad -(17)$$

in which case:

$$\underline{\overline{W}} \cdot \underline{\nabla} \times \underline{\overline{W}} = k \underline{\overline{W}} \cdot \underline{\overline{W}} \quad -(18)$$

From eqns (9) and (18):

$$\boxed{\underline{\nabla} \cdot \underline{A} \times \underline{\overline{W}} = \frac{k \underline{\overline{W}} \cdot \underline{\overline{W}}}{r^{(0)}}} \quad -(19)$$

This is the condition for vanishing charge current density of magnetism if it is assumed that $\underline{\overline{W}}$ is a Beltrami field:

$$\underline{\nabla} \times \underline{\overline{W}} = k \underline{\overline{W}} \quad -(20)$$

Units Check

$A^{(0)} = mT$, $\underline{A} = mT$; $\overline{W}^{(0)} = m^3 T$; $\underline{\overline{W}} = m^2 T$,
$k = m^{-1}$, $r^{(0)2} = m^2$, so

$$m^{-1} mT m^2 T = \frac{m^{-1} m^6}{m^2} T^2 = m^2 T \checkmark$$

Under the assumption:

$$\frac{k \underline{\overline{W}} \cdot \underline{\overline{W}}}{r^{(0)}} = \text{constant} := x \quad -(21)$$

then

$$\boxed{\underline{\nabla} \cdot \underline{A} \times \underline{\overline{W}} = x} \quad -(22)$$

means that there is no magnetic charge / current density.

Finally denote:

$$\underline{\sigma} = \underline{A} \times \underline{\bar{W}} \quad -(23)$$

then

$$\oint_S \underline{\sigma} \cdot \underline{n}\, dA = \int_V \underline{\nabla} \cdot \underline{\sigma}\, dV \quad -(24)$$

So

$$\int_V \underline{\nabla} \cdot \underline{A} \times \underline{\bar{W}}\, dV = \oint_S \underline{A} \times \underline{\bar{W}} \cdot \underline{n}\, dA$$

$$= \int x\, dV \quad -(25)$$

Therefore this is the condition for zero magnetic charge / current density if it is assumed that:

$$\underline{\nabla} \times \underline{\bar{W}} = k\, \underline{\bar{W}} \quad -(26)$$

The general condition is eq. (9):

$$\boxed{\underline{\nabla} \cdot \underline{A} \times \underline{\bar{W}} = \frac{1}{r^{(0)}}\, \underline{\bar{W}} \cdot \underline{\nabla} \times \underline{\bar{W}}} \quad -(27)$$