048(5): Force and Binet Equations for a Uniform Field

The force equation is worked out from the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 - \frac{\Omega^2 r^2}{2} \right) + \frac{mMG}{r} \quad -(1)$$

and the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(2)$$

From eqs. (1) and (2):

$$\boxed{m \left(\ddot{r} - r \left(\dot{\theta}^2 - \frac{\Omega^2}{2} \right) \right) = -\frac{\partial U}{\partial r} = F(r)} \quad -(3)$$

Here:

$$U = -\frac{mMG}{r}, \quad F = -\frac{mMG}{r^2} \quad -(4)$$

Eq. (3) is the Lorentz force equation for an orbital precession and for a uniform gravitomagnetic field.

In eq. (3):

$$\dot{\theta} = \frac{L}{mr^2} \quad -(5)$$

and if there is no precession:

$$\Omega = 0. \quad -(6)$$

Note carefully that if there is no precession Eq. (3) reduces to the Leibnitz equation. Eq. (3) also contains the observed precession frequency Ω.

To transform eq. (3) into a Binet equation

2) use:

$$u = \frac{1}{r} \quad -(7)$$

to find that:

$$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{mr^2}{r^2 L}\frac{dr}{dt} \quad -(8)$$

so that:

$$\frac{du}{d\theta} = -\frac{m}{L}\frac{dr}{dt}. \quad -(9)$$

Furthermore:

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = -\frac{m}{L}\frac{d}{d\theta}\frac{dr}{dt} \quad -(10)$$

where:

$$\frac{d}{d\theta}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{dr}{dt}\right)\frac{dt}{d\theta}. \quad -(11)$$

Therefore:

$$\frac{d^2u}{d\theta^2} = -\frac{m^2r^2}{L^2}\frac{d^2r}{dt^2} \quad -(12)$$

This means that:

$$m\ddot{r} - r\dot{\theta}^2 = F(r) + \frac{mr\Omega^2}{2} \quad -(13)$$

transforms to:

$$\boxed{F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) - \frac{m\Omega^2 r}{2}} \quad -(14)$$

which is the precessional Binet equation.

As shown in Note 348(1), the orbit in general is:

$$r = \frac{\alpha}{1+\epsilon\cos(x(\theta)\theta)} \quad -(15)$$

where

$$x(\theta) = 1 + \frac{1}{2\theta}\left(\frac{m\alpha_0^2\Omega}{L}\right)^2 \int \frac{d\theta}{(1+\epsilon\cos\theta)^4} \quad -(16)$$

and the force corresponding to this orbit is:

$$F(r) = m\ddot{r} - r\dot{\theta}^2 - \frac{mr}{2}\Omega_0^2 . \quad -(17)$$
