348(2): Development of the Orbital Equation from the Lorentz Force Equation.

The Lorentz force equation derived in UFT347 is:

$$\underline{F} = m\ddot{\underline{r}} = -\frac{mMG}{r^2}\underline{e}_r + m\frac{d\underline{v}_g}{dt} - m\dot{\underline{r}} \times \underline{\Omega}_g . \quad -(1)$$

In note 348(1) it was shown that this equation corresponds to the orbit:

$$r = \frac{\alpha}{1+\epsilon\cos(x(\theta)\theta)} \quad -(2)$$

which is a precessing orbit. The factor $x(\theta)$ depends on θ. In eq. (1):

$$m\dot{\underline{r}} = \underline{p} + m\underline{v}_g \quad -(3)$$

is the canonical momentum. In the absence of $\underline{v}_g$ eq. (1) reduces to the Leibnitz equation:

$$\underline{F} = m(\ddot{r} - r\dot{\theta}^2)\underline{e}_r = -\frac{mMG}{r^2}\underline{e}_r . \quad -(4)$$

In eq. (4),

$$\ddot{\underline{r}} = \frac{d\dot{\underline{r}}}{dt} = \frac{d\underline{v}}{dt} \quad -(5)$$

Therefore eq. (4) is:

$$\ddot{r} - r\dot{\theta}^2 = F(r), \quad -(6)$$

which can be transformed into the Binet Equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \quad -(7)$$

1) In the first approximation, eqns (2) and (7) may be used to find the force $F(r)$.

However, it is already known that the force needed for the orbit (2) is the Lorentz force (1). It would be interesting to find the Binet equation corresponding to the Lorentz force. It is now known that it is the Lorentz force that gives any precessing orbit. Eq. (1) has the great advantage of including any observed precession, which is

$$\Omega = \frac{1}{2} |\Omega_g| \quad -(8)$$

The Leibnitz equation (6) gives the non precessing orbit:

$$r = \frac{\alpha}{1+\epsilon\cos(\theta)} \quad -(9)$$

Therefore the extra terms in eq. (1) give precession of any kind.

From eqs. (1) and (3):

$$\underline{F} = m\ddot{\underline{r}} = m\left(\frac{d\underline{v}}{dt} + \frac{d\underline{v}_g}{dt}\right) \quad -(10)$$

$$= -\frac{mMG}{r^2}\underline{e}_r + m\frac{d\underline{v}_g}{dt} - m\dot{\underline{r}} \times \underline{\Omega}_g$$

Consider:

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \frac{\partial X}{\partial t}\frac{\partial v_x}{\partial X} + \dots \quad -(11)$$

To first order:

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \frac{\partial X}{\partial t}\frac{\partial v_x}{\partial X}\underline{i} + \frac{\partial Y}{\partial t}\frac{\partial v_y}{\partial Y}\underline{j} + \frac{\partial Z}{\partial t}\frac{\partial v_z}{\partial Z}\underline{k} \quad -(12)$$

$$= \frac{\partial \underline{v}}{\partial t} + \left(\underline{\dot{R}} \cdot \underline{\nabla}\right)\underline{v}$$

where

$$\underline{\dot{R}} = \frac{\partial X}{\partial t}\underline{i} + \frac{\partial Y}{\partial t}\underline{j} + \frac{\partial Z}{\partial t}\underline{k} \quad -(13)$$

It follows that:

$$\frac{d\underline{v}}{dt} - \frac{\partial \underline{v}}{\partial t} = \left(\underline{\dot{R}} \cdot \underline{\nabla}\right)\underline{v} \quad -(14)$$

Therefore eq. (10) becomes:

$$\underline{F} = m\frac{d\underline{v}}{dt} = -\frac{mMG}{r^2}\underline{e}_r - \left(\underline{\dot{R}} \cdot \underline{\nabla}\right)\underline{v}_g - m\underline{\dot{r}} \times \underline{\Omega}_g \quad -(15)$$

$$= m\left(\ddot{R} - r\dot{\theta}^2\right)\underline{e}_r$$

Here:

$$\underline{R} = X\underline{i} + Y\underline{j} + Z\underline{k} = R\underline{e}_r \quad -(16)$$

$$\underline{v} = \frac{d\underline{R}}{dt} = \dot{R}\underline{e}_r + R\dot{\theta}\underline{e}_\theta = \underline{\dot{R}} \quad -(17)$$

and

$$\underline{\ddot{R}} = \left(\ddot{R} - R\dot{\theta}^2\right)\underline{e}_r \quad -(18)$$

for a planar orbit.

4) If it is assumed that:

$$\underline{\nabla} \cdot \underline{v}_g = 0 \quad -(19)$$

then:

$$\underline{F} = m(\ddot{R} - r\dot{\theta}^2)\underline{e}_r = -\frac{mMG}{r^2}\underline{e}_r - m\dot{\underline{r}} \times \underline{\Omega}_g \quad -(20)$$

where the canonical momentum is:

$$m\dot{\underline{r}} = m(\underline{v} + \underline{v}_g) = m(\dot{\underline{R}} + \underline{v}_g). \quad -(21)$$

The Precession Term

This is the extra force:

$$\underline{F}_p = -m\dot{\underline{r}} \times \underline{\Omega}_g \quad -(22)$$

$$= -m(\dot{\underline{R}} + \underline{v}_g) \times \underline{\Omega}_g$$

$$= -m(\dot{R}\underline{e}_r + R\dot{\theta}\underline{e}_\theta + \underline{v}_g) \times \underline{\Omega}_g$$

By definition:

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_g, \quad -(23)$$

and for a uniform $\underline{\Omega}_g$:

$$\underline{v}_g = \frac{1}{2}\underline{\Omega}_g \times \underline{R}. \quad -(24)$$

Force in the $\underline{e}_r$ direction can only be obtained from:

$$\underline{F}_p = -m(R\dot{\theta}\underline{e}_\theta + \underline{v}_g) \times \underline{\Omega}_g \quad -(25)$$

because $\underline{e}_r \times \underline{\Omega}_g$ cannot produce an $\underline{e}_r$ component.

5) Therefore:

$$\underline{F} = m\left(\ddot{R} - R\dot{\theta}^2\right)\underline{e}_r = -\frac{mMG}{r^2}\,\underline{e}_r - mR\dot{\theta}\left(\underline{e}_\theta \times \underline{\Omega}_g\right) - \frac{m}{2}\left(\underline{\Omega}_g \times \underline{R}\right) \times \underline{\Omega}_g \quad -(26)$$

Now use the vector identity:

$$\underline{F} \times \left(\underline{G} \times \underline{H}\right) = \underline{G}\left(\underline{F}\cdot\underline{H}\right) - \underline{H}\left(\underline{F}\cdot\underline{G}\right) \quad -(27)$$

$$-(28)$$

to find that:

$$\left(\underline{\Omega}_g \times \underline{R}\right) \times \underline{\Omega}_g = \underline{R}\left(\underline{\Omega}_g \cdot \underline{\Omega}_g\right) - \underline{\Omega}_g\left(\underline{\Omega}_g \cdot \underline{R}\right)$$

If the velocity $\underline{v}_g$ is in the plane of the orbit, then:

$$\underline{\Omega}_g = \Omega_g\,\underline{k}. \quad -(29)$$

Using:

$$\underline{R} = R\,\underline{e}_r \quad -(30)$$

the vector $\underline{R}$ is in the plane of the orbit, so:

$$\underline{\Omega}_g \cdot \underline{R} = 0 \quad -(31)$$

Finally, using:

$$\underline{e}_\theta \times \underline{k} = \underline{e}_r \quad -(32)$$

it is found that:

$$F = m\left(\ddot{R} - R\dot{\theta}^2\right) = -\frac{mMG}{R^2} - mR\dot{\theta}\,\Omega_g - \frac{m}{2}\,\Omega_g^2 R \quad -(33)$$

). So the Leibnitz equation becomes:

$$\ddot{R} - R\dot{\theta}^2 = -\frac{MG}{R^2} - \dot{\theta}\Omega_g - \frac{1}{2}\Omega_g^2 R \quad -(34)$$

The observed precession is:

$$\Omega = \frac{1}{2}\Omega_g \quad -(35)$$

So:

$$\ddot{R} - R\dot{\theta}^2 = -\frac{MG}{R^2} - 2\Omega\dot{\theta} - 2\Omega^2 R \quad -(36)$$

The Leibnitz equation with precession is therefore:

$$\boxed{\ddot{R} - R\dot{\theta}^2 = -\frac{MG}{R^2} - 2\Omega\left(\dot{\theta} + \Omega R\right)} \quad -(37)$$

This produces the orbit:

$$R = \frac{\alpha}{1 + \epsilon\cos(x(\theta)\theta)} \quad -(38)$$

which is a precessing orbit, with precession frequency Ω. Therefore the precessional problem has been worked out exactly within the given approximations. The next stage is to deduce the Binet equation from eq. (37). Note carefully that the inviscid spacetime is defined by:

$$\left.\begin{aligned} \underline{\Omega}_g &= \underline{\nabla} \times \underline{v}_g, \\ \underline{\nabla} \cdot \underline{v}_g &= 0 \end{aligned}\right\} \quad -(39)$$

[illegible] no. 471 ff.)