348(1): Precessing Ellipse from the Minimal Prescription

As in previous notes the Hamiltonian from the minimal prescription is:

$$H = \frac{p^2}{2m} + U(r) = \frac{1}{2}m\left(v^2 + v_g^2\right) + U(r) + \Omega L \quad -(1)$$

where Ω is the precession frequency and where L is the magnitude of the angular momentum. The minimal prescription is considered to be:

$$\underline{p} \rightarrow \underline{p} + m\underline{v}_g. \quad -(2)$$

As in previous work:

$$v_g^2 = \Omega^2 r^2 \quad -(3)$$

so:

$$H = \frac{1}{2}m\left(v^2 + \Omega^2 r^2\right) + U(r) + \Omega L \quad -(4)$$

where:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(5)$$

Therefore the Hamiltonian is:

$$H = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\left(\frac{d\theta}{dt}\right)^2 + \Omega^2\right)\right) + U(r) + \Omega L \quad -(6)$$

i.e.

$$H_1 = H - \Omega L = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta_1}{dt}\right)^2\right) + U(r) \quad -(7)$$

which leads to the orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta_1} \quad -(8)$$

where

$$\left(\frac{d\theta_1}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 + \Omega^2. \quad -(9)$$

/ Denote: $$\omega_1 = \frac{d\theta_1}{dt}, \quad \omega = \frac{d\theta}{dt} \quad -(10)$$

then:
$$\omega_1^2 = \omega^2 + \Omega^2 \quad -(11)$$

and
$$\omega_1^2 = \omega^2\left(1 + \left(\frac{\Omega}{\omega}\right)^2\right) \quad -(12)$$

so
$$\omega_1 = \omega\left(1 + \left(\frac{\Omega}{\omega}\right)^2\right)^{1/2} \quad -(13)$$

If
$$\Omega \ll \omega \quad -(14)$$

then
$$\omega_1 \sim \omega\left(1 + \frac{1}{2}\left(\frac{\Omega}{\omega}\right)^2\right)^{1/2} \quad -(15)$$

i.e.
$$\frac{d\theta_1}{dt} = \left(1 + \frac{1}{2}\left(\frac{\Omega}{\omega}\right)^2\right)\frac{d\theta}{dt} \quad -(16)$$

So
$$d\theta_1 = \left(1 + \frac{1}{2}\left(\frac{\Omega}{\omega}\right)^2\right) d\theta \quad -(17)$$

and
$$\boxed{\theta_1 = \int\left(1 + \frac{1}{2}\left(\frac{\Omega}{\omega}\right)^2\right) d\theta} \quad -(18)$$

In Eq. (18), ω is the angular velocity corresponding to the Hamiltonian:
$$H = \frac{p_0^2}{2m} + U(r) \quad -(19)$$

3)

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + U(r)$$

This angular velocity is:

$$\omega = \frac{L_0}{mr^2} \quad -(20)$$

where L_0 is a constant of motion. In Eq. (20):

$$r = \frac{\alpha_0}{1 + \epsilon_0 \cos\theta} \quad -(21)$$

Therefore:

$$\frac{1}{\omega^2} = \frac{m^2 r^4}{L_0^2} = \frac{m^2 \alpha_0^4}{L_0^2 (1 + \epsilon_0 \cos\theta)^4} \quad -(22)$$

So:

$$\theta_1 = \int \left(1 + \frac{m^2 \alpha_0^4 \Omega^2}{2 (1 + \epsilon_0 \cos\theta)^4 L_0^2} \right) d\theta \quad -(23)$$

and the orbit is:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta_1} \quad -(24)$$

In previous work it was assumed that:

$$\theta_1 = x\theta \quad -(25)$$

In these equations:

$$L = mr^2 \frac{d\theta_1}{dt} = \text{constant} \quad -(26)$$

$$L_0 = mr^2 \frac{d\theta}{dt} = \text{constant} \quad -(27)$$

$$\alpha = \frac{L^2}{m^2 M G}, \quad \alpha_0 = \frac{L_0^2}{m^2 M G}, \quad -(28)$$

$$\epsilon = \left(1 + \frac{2 H_1 L^2}{m^3 M^2 G^2}\right)^{1/2} \quad -(29)$$

$$\epsilon_0 = \left(1 + \frac{2 H L_0^2}{m^3 M^2 G^2}\right)^{1/2} \quad -(30)$$

Eq. (23) is:

$$\theta_1 = \theta + \frac{m^2 \alpha_0^4 \Omega^2}{2 L_0^2} \int \frac{d\theta}{(1 + \epsilon_0 \cos\theta)^4} \quad -(31)$$

The integral can be evaluated by computer algebra and the orbit (24) plotted.

If it is assumed that:

$$\theta_1 = x\theta \quad -(32)$$

as in previous work then:

$$\theta_1 = \int \left(1 + \frac{1}{2}\left(\frac{\Omega}{\omega}\right)^2\right) d\theta = x\theta \quad -(33)$$

and

$$-(34)$$

$$x\theta = \theta + \frac{m^2 \alpha_0^4 \Omega^2}{2 L_0^2} \int \frac{d\theta}{(1 + \epsilon_0 \cos\theta)^4} \quad -(35)$$

So it follows that:

5)

$$x = 1 + \frac{1}{2\theta}\left(\frac{m\alpha_0^2\Omega}{L_0}\right)^2 \int \frac{d\theta}{(1+\epsilon_0\cos\theta)^4} \quad -(36)$$

The orbit is then:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(37)$$

which can be plotted using computer algebra. This should give some very interesting results, all stemming from the simple minimal prescription:

$$\underline{p} \rightarrow \underline{p} + m\underline{v}_g \quad -(38)$$

with observed precession Ω.