348(4): Lagrangian for a Uniform Gravitomagnetic Field

In this case:

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_g \quad -(1)$$

and

$$\underline{v}_g = \frac{1}{2} \underline{\Omega}_g \times \underline{r} \quad -(2)$$

As in Note 347(2):

$$v_g^2 = \frac{1}{4} \underline{\Omega}_g \times \underline{r} \cdot \underline{\Omega}_g \times \underline{r}$$

$$= \frac{1}{4} \left(\Omega_g^2 r^2 - (\underline{\Omega}_g \cdot \underline{r})(\underline{\Omega}_g \cdot \underline{r}) \right) \quad -(3)$$

If the gravitomagnetic field is perpendicular to the plane of the orbit, then:

$$v_g^2 = \frac{1}{4} \Omega_g^2 r^2 \quad -(4)$$

The lagrangian is:

$$\mathcal{L} = \frac{1}{2m} (\underline{p} + m\underline{v}_g) \cdot (\underline{p} + m\underline{v}_g) - (\underline{p} + m\underline{v}_g) \cdot \underline{v}_g - U(r)$$

$$= \frac{1}{2m} \left(p^2 + m(\underline{v}_g \cdot \underline{p} + \underline{p} \cdot \underline{v}_g) + m^2 v_g^2 \right) - \underline{p} \cdot \underline{v}_g - m v_g^2 - U(r)$$

$$= \frac{p^2}{2m} - \frac{1}{2} m v_g^2 - U(r) \quad -(5)$$

$$= \frac{1}{2} m (v^2 - v_g^2) - U(r)$$

From eqs. (4) and (5):

2) $$\mathcal{L} = \frac{1}{2}m\left(v^2 - \frac{\Omega_g^2 r^2}{4}\right) - U(r) \quad - (6)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 - \frac{\Omega_g^2 r^2}{4}\right) - U(r).$$

The Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad - (7)$$

So the conserved angular momentum is:

$$L = mr^2\dot{\theta}. \quad - (8)$$

So the precessional Leibnitz equation is, in general, for any $\underline{\Omega}_g$: $\quad - (9)$

$$\underline{F} = m\frac{d\underline{v}}{dt} = -\frac{mMG}{r^2}\underline{e}_r - (\underline{\dot{R}}\cdot\underline{\nabla})\underline{v}_g - m\underline{\dot{r}}\times\underline{\Omega}_g$$

together with eq. (8) if $\underline{\Omega}_g$ is uniform. When $\underline{\Omega}_g$ is uniform it is easier to derive the force equation from the Lagrangian (6) and the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}. \quad - (10)$$

This will be done in the next note.