348(3): Calculation of the Constant Angular Momentum from the Precessional Lagrangian

With the minimal prescription:

$$\underline{p} \rightarrow \underline{p} + m\underline{v}_g \quad -(1)$$

the precessional Lagrangian is:

$$\mathcal{L} = \frac{1}{2m}(\underline{p} + m\underline{v}_g)\cdot(\underline{p} + m\underline{v}_g) - (\underline{p} + m\underline{v}_g)\cdot\underline{v}_g - U(r) \quad -(2)$$

and gives the Lorentz force equation.

The Hamiltonian is:

$$H = \frac{1}{m}(\underline{p} + m\underline{v}_g)\cdot\underline{p} - \mathcal{L} \quad -(3)$$

$$= \frac{1}{m}\underline{p}\cdot(\underline{p} + m\underline{v}_g) + \underline{v}_g\cdot(\underline{p} + m\underline{v}_g) - \frac{1}{2m}(\underline{p} + m\underline{v}_g)\cdot(\underline{p} + m\underline{v}_g) + U(r)$$

$$= \frac{1}{m}\left(\underline{p}\cdot(\underline{p} + m\underline{v}_g) + m\underline{v}_g\cdot(\underline{p} + m\underline{v}_g)\right) - \frac{1}{2m}(\underline{p} + m\underline{v}_g)\cdot(\underline{p} + m\underline{v}_g) + U(r)$$

$$= \frac{1}{2}(\underline{p} + m\underline{v}_g)\cdot(\underline{p} + m\underline{v}_g) + U(r)$$

Q.E.D.

For a planar orbit with:

$$\underline{r} = \underline{r}(t) \quad -(4)$$

the velocities are defined by:

2) $\underline{v} = \frac{d\underline{r}}{dt}$; $\underline{v}_g = \frac{d\underline{r}_g}{dt}$ — (5)

So: $\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta$ — (6)

$\underline{v}_g = \dot{r}_g\,\underline{e}_r + r_g\dot{\theta}\,\underline{e}_\theta$ — (7)

So: $\underline{v} + \underline{v}_g = (\dot{r} + \dot{r}_g)\underline{e}_r + (r + r_g)\dot{\theta}\,\underline{e}_\theta$ — (8)

and $(\underline{v} + \underline{v}_g)\cdot(\underline{v} + \underline{v}_g) = (\dot{r} + \dot{r}_g)^2 + (r + r_g)^2\dot{\theta}^2$. — (9)

Similarly: $(\underline{v} + \underline{v}_g)\cdot\underline{v}_g = (\dot{\underline{r}} + \dot{\underline{r}}_g)\cdot\dot{\underline{r}}_g$

$= \left((\dot{r} + \dot{r}_g)\underline{e}_r + (r + r_g)\dot{\theta}\,\underline{e}_\theta\right)\cdot\left(\dot{r}_g\,\underline{e}_r + r_g\dot{\theta}\,\underline{e}_\theta\right)$

$= (\dot{r} + \dot{r}_g)\dot{r}_g + (r + r_g)r_g\dot{\theta}^2$ — (10)

So the lagrangian is:

$\mathcal{L} = \frac{1}{2}m\left((\dot{r} + \dot{r}_g)^2 + (r + r_g)\dot{\theta}^2\right) - U(r)$

$-m\left((\dot{r} + \dot{r}_g)\dot{r}_g + (r + r_g)r_g\dot{\theta}^2\right)$ — (11)

The rotational Euler lagrange equation is:

$\frac{\partial\mathcal{L}}{\partial\theta} = 0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$ — (12)

The conserved angular momentum is therefore:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \text{constant.} \quad -(13)$$

So:

$$L = m\left((r+r_g)^2\dot{\theta} - 2(r+r_g)\,r_g\,\dot{\theta}\right)$$

$$= m\dot{\theta}\left((r+r_g)^2 - 2r_g(r+r_g)\right)$$

$$= m\dot{\theta}\,(r+r_g)(r+r_g-2r_g),$$

$$\boxed{L = m\dot{\theta}\,(r+r_g)(r-r_g)}. \quad -(14)$$

In Note 348(2) it was shown that the Leibnitz equation with precession is:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{MG}{r^2} - 2\Omega(\dot{\theta} + \Omega r). \quad -(15)$$

where

$$\Omega = \frac{1}{2}|\underline{\Omega}_g| \quad -(16)$$

and

$$\underline{\Omega}_g = \underline{\nabla} \times \underline{v}_g \quad -(17)$$

Eqs. (14) to (17) are the complete dynamical equations of the system.