346(2): Development of the General Precession Law

The general precession law is:

$$\Omega = \frac{G}{2c^2 r^3} \left| \nabla \times (\underline{L} \times \underline{r}) \right| \quad -(1)$$

where the angular momentum $\underline{L}$ is defined with respect to coordinates in a rotating mass distribution. Consider the well known principal axes of inertia, then:

$$\underline{L} = I \, \underline{\omega} \quad -(2)$$

where $\underline{\omega}$ is the angular velocity vector and where I is the moment of inertia about a principal axis of rotation. For example, in a rotating sphere:

$$\underline{L} = \frac{2}{5} M R^2 \omega \, \underline{k} \quad -(3)$$

where R is the radius of the sphere, M its mass, and ω its angular velocity about $\underline{k}$.

In general the moment of inertia is defined as

$$I = \int \rho r^2 dV \quad -(4)$$

where ρ is the mass density. For a disk:

$$I = \int \rho r^2 dA \quad -(5)$$

where dA is the area element:

$$dA = r \, dr \, d\theta \quad -(6)$$

Therefore:

$$I(\text{disk}) = \int_0^{2\pi} \int_0^R \rho r^3 \, dr \, d\theta = 2\pi \rho \frac{R^4}{4} \quad -(7)$$

2) The mass density per unit area is:

$$\rho = \frac{m}{\pi R^2} \quad -(8)$$

So $$I(\text{disk}) = \frac{mR^2}{2} \quad -(9)$$

In order to calculate the moment of inertia of a sphere use:

$$dI = \frac{1}{2} r^2 \, dm \quad -(10)$$

and $$dm = \rho \, dV; \quad dV = \pi r^2 \, dx \quad -(11)$$

so $$dm = \rho \pi r^2 \, dx, \quad dI = \frac{1}{2} \rho \pi r^4 \, dx \quad -(12)$$

Now use $r^2 = R^2 - x^2$, so

$$dI = \frac{1}{2} \rho \pi (R^2 - x^2)^2 \, dx \quad -(13)$$

Therefore:

$$I = \frac{1}{2} \rho \pi \int_{-R}^{R} (R^2 - x^2)^2 \, dx \quad -(14)$$

$$= \frac{1}{2} \rho \pi \frac{16}{15} R^5$$

where

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3} \pi R^3} \quad -(15)$$

So $$I = \frac{2}{5} M R^2 \quad -(16)$$

Therefore in general the angular momentum of a localized mass distribution such as the sun is:

$$\underline{L} = \int \underline{R} \times \underline{J}(\underline{R})\, d^3 R,$$

$$\underline{L} = \left(\int \rho R^2 \, d\bar{V} \right) \underline{\omega} \quad -(17)$$

$$= \underline{I}\, \underline{\omega} .$$

Therefore:

$$\Omega = \frac{IG}{2c^2 r^3} \left| \underline{\nabla} \times (\underline{\omega} \times \underline{r}) \right| \quad -(18)$$

where

$$I = \int \rho R^2 \, d\bar{V} . \quad -(19)$$

If it is assumed that:

$$\underline{\omega} = \omega \underline{k} \quad -(20)$$

then

$$\Omega = \frac{IG\omega}{2c^2 r^3} \left| \underline{\nabla} \times (\underline{k} \times \underline{r}) \right| \quad -(21)$$

If the orbit is in the plane $\perp \underline{k}$ then:

$$\underline{r} = X \underline{i} + Y \underline{j} . \quad -(22)$$

If the orbit is inclined relative to $\underline{k}$ then:

$$\underline{r} = X \underline{i} + Y \underline{j} + Z \underline{k} . \quad -(23)$$

All solar system orbits are inclined relative to the spin axis $\underline{k}$ of the sun.