343(1): Thomas and de Sitter Precession and Relativistic Momentum

Consider the infinitesimal line element of ECE2 relativity:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

$$= (c^2 - v_N^2)\, dt^2$$

where

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(2)$$

Thomas precession is calculated from a rotation:

$$d\theta \rightarrow d\theta + \omega\, dt \quad -(3)$$

where

$$v = r\omega \quad -(4)$$

is the tangential linear velocity of the rotation.

It follows that:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 (d\theta + \omega\, dt)^2 \quad -(4)$$

and that the Newtonian velocity is changed to:

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \frac{d}{dt^2}(d\theta + \omega\, dt)^2$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt} + \omega\right)^2 \quad -(5)$$

The frame rotation leading to the Thomas precession can be described by rotating plane polar coordinates (r, θ'), where

$$\theta_1 = \theta + \omega t \quad -(6)$$

in which

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta_1}{dt}\right)^2 \quad -(7)$$

) In a lagrangian analysis the two variables are r and θ', and the two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \quad -(8)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(9)$$

The lagrangian is:

$$\mathcal{L}_0 = \frac{1}{2} m v_N^2 - U \quad -(10)$$

where U is the potential energy if the extra rotation is regarded as non-relativistic. However for self consistency, the relativistic lagrangian must be used:

$$\mathcal{L} = -\frac{mc^2}{\gamma_1} - U \quad -(11)$$

because the starting point of the analysis is a relativistic infinitesimal line element, eq. (1). Eq. (11) is:

$$\mathcal{L} = -mc^2\left(1 - \frac{v_N^2}{c^2}\right)^{1/2} - U \quad -(12)$$

$$= -mc^2\left(1 - \frac{1}{c^2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta_1}{dt}\right)^2\right)\right)^{1/2} - U$$

where

$$U = -\frac{mMG}{r} \quad -(13)$$

and

$$\dot{\theta}_1 = \frac{d\theta_1}{dt} \quad -(14)$$

3) As in Note 324(4), if

$$f := 1 - \frac{1}{c^2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta_1}{dt}\right)^2\right)^{1/2} - U \quad -(15)$$

then

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = -\frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \dot{\theta}_1} = \frac{mc^2}{2} f^{-1/2} \cdot \frac{2r^2\dot{\theta}_1}{c^2} \quad -(16)$$

$$= \gamma_1 m r^2 \dot{\theta}_1 := L_1$$

where

$$\frac{dL_1}{dt} = 0 \quad -(17)$$

The relativistic angular momentum in the rotating frame (r, θ_1) is therefore:

$$L_1 = \gamma_1 m r^2 \dot{\theta}_1 \quad -(18)$$

and is a constant of motion. The Newtonian velocity (5) can therefore be written as:

$$v_{N1}^2 = \frac{L_1^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta_1}\right)^2\right) \quad -(19)$$

in the rotating coordinate system (r, θ_1). In the static coordinate system:

$$v_N^2 = \frac{L^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right) \quad -(20)$$

In this equation both v_{N1} and v_N are defined by

) relativistic line elements, eqs. (4) and (1) respectively.
Both L^2 and L_1^2 are constants of motion.
For the static frame, the relativistic velocity is:

$$v^2 = \gamma^2 v_N^2 = \frac{v_N^2}{1 - \frac{v_N^2}{c^2}} \quad -(21)$$

For the rotating frame, the relativistic velocity is:

$$v^2 = \gamma_1^2 v_{N1}^2 = \frac{v_{N1}^2}{1 - \frac{v_{N1}^2}{c^2}} \quad -(22)$$

Eqs. (21) and (22) generate two Evans Eckardt theorems. Eq. (21) relates the precessing elliptical orbit to the static elliptical orbit. Eq. (22) gives the geodetic precession, or de Sitter precession. This will be shown in the next note.

As in UFT 110, the Thomas precession is worked out using:

$$ds^2 = (c^2 - r^2\omega^2)dt^2 - 2\omega r^2 d\theta dt - dr^2 - r^2 d\theta^2$$

$$:= \left(1 - \frac{v^2}{c^2}\right)\left(c^2 dt^2 - 2r^2 \Omega d\theta dt\right) - dr^2 - r^2 d\theta^2 \quad -(23)$$

where the angular velocity in the rotating frame is:

$$\Omega = \omega\left(1-\frac{v^2}{c^2}\right)^{-1} \quad -(24)$$

and where the infinitesimal of time in the rotating frame is:

$$dt_1 = \left(1-\frac{v^2}{c^2}\right)^{1/2} dt \quad -(25)$$

This gives the Thomas phase shift:

$$\alpha = \Omega\, dt_1 - \omega\, dt$$

$$= 2\pi\left(\left(1-\frac{v^2}{c^2}\right)^{1/2} - 1\right) \quad -(26)$$

for a rotation of 2π radians.

This occurs in addition to the precession of the orbit.