343(2): Thomas Precession in the Newtonian Limit

The rotation of the frame in the Newtonian limit is defined by:

$$\theta_1 = \theta + \omega_\theta t \quad -(1)$$

and

$$\omega_1 = \frac{d\theta_1}{dt} = \frac{d\theta}{dt} + \omega_\theta \quad -(2)$$

The frame is rotated by a constant angular velocity ω_θ. Therefore the plane polar coordinate system (r, θ) is changed to (r, θ_1). The lagrangian becomes:

$$\mathcal{L}_1 = \frac{1}{2} m v_1^2 - U \quad -(3)$$

where

$$v_1^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta_1}{dt}\right)^2 \quad -(4)$$

The Euler Lagrange equations become:

$$\frac{\partial \mathcal{L}_1}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \dot{\theta}_1} \; ; \; \frac{\partial \mathcal{L}_1}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \dot{r}} \quad -(5)$$

The conserved angular momentum becomes:

$$L_1 = m r^2 \frac{d\theta_1}{dt} \quad -(6)$$

Therefore

$$L_1 = m r^2 \left(\frac{d\theta}{dt} + \omega_\theta\right) \quad -(7)$$

$$= L + m r^2 \omega_\theta$$

$$L_1 = L + m r^2 \omega_\theta \quad -(8)$$

Note that ω_θ is defined to be constant, so both

2) L_1 and L are constants of motion, Q.E.D

The Hamiltonian is:

$$H_1 = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L_1^2}{mr^2} + U(r) \quad -(9)$$

where

$$U(r) = -\frac{mMG}{r} \quad -(10)$$

so

$$\frac{dr}{dt} = \left(\frac{2}{m}(H_1 - U) - \frac{L_1^2}{m^2r^2}\right)^{1/2} \quad -(11)$$

$$= \frac{dr}{d\theta_1}\frac{d\theta_1}{dt} = \frac{L_1}{mr^2}\frac{dr}{d\theta_1}$$

so

$$\frac{dr}{d\theta_1} = \frac{mr^2}{L_1}\left(\frac{2}{m}(H_1 - U) - \frac{L_1^2}{m^2r^2}\right)^{1/2} \quad -(12)$$

and

$$\frac{d\theta_1}{dr} = \frac{L_1}{mr^2}\left(\frac{2}{m}(H_1 - U) - \frac{L_1^2}{m^2r^2}\right)^{-1/2} \quad -(13)$$

so

$$\theta_1(r) = \int \frac{L_1\, dr}{r^2\left(2m\left(H_1 - U - \frac{L_1^2}{2mr^2}\right)\right)^{1/2}} \quad -(14)$$

This gives

$$r = \frac{\alpha_1}{1 + \epsilon_1 \cos\theta_1} \quad -(15)$$

where

$$\alpha_1 = \frac{L_1^2}{mM^2G} \quad -(16)$$

and $$\epsilon_1 = \left(1 + \frac{2 H_1 L^2}{m^3 M^2 G^2} \right)^{1/2} \quad -(17)$$

Therefore the Thomas effect on the Newtonian level results in the conic section:

$$\boxed{r = \frac{\alpha_1}{1 + \epsilon_1 \cos(\theta + \omega_\theta t)}} \quad -(18)$$

Finally define:

$$x\theta := \theta + \omega_\theta t \quad -(19)$$

so

$$x := 1 + \frac{\omega_\theta t}{\theta} \quad -(20)$$

$$= 1 + \frac{v_\theta r t}{\theta}$$

where

$$\omega_\theta = v_\theta r \quad -(21)$$

so

$$r = \frac{\alpha_1}{1 + \epsilon_1 \cos(x\theta)} \quad -(22)$$

where

$$x = 1 + \frac{\omega_\theta t}{\theta} \quad -(23)$$

where ω_θ is a constant. This result can be graphed.

In the next note the theory can be put on a relativistic level.