340(6): Cyclic Symmetry and $\underline{B}^{(3)}$ Field from the $\underline{W}$ Potentials

Using the result: $\underline{W}^a = \underline{A}^a$ — (1)

it follows that:

$$\underline{B}^{(1)*} = \underline{\nabla} \times \underline{W}^{(1)*} - \frac{i}{W^{(0)}} \underline{W}^{(2)} \times \underline{W}^{(3)} \quad -(2)$$

$$\underline{B}^{(2)*} = \underline{\nabla} \times \underline{W}^{(2)*} - \frac{i}{W^{(0)}} \underline{W}^{(3)} \times \underline{W}^{(1)} \quad -(3)$$

$$\underline{B}^{(3)*} = \underline{\nabla} \times \underline{W}^{(3)*} - \frac{i}{W^{(0)}} \underline{W}^{(1)} \times \underline{W}^{(2)} \quad -(4)$$

in the complex circular basis:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i \underline{e}^{(3)*} \quad -(5)$$

et cyclicum

The Aharonov Bohm vacuum is therefore defined by:

$$\underline{\nabla} \times \underline{W}^{(1)*} = \frac{i}{W^{(0)}} \underline{W}^{(2)} \times \underline{W}^{(3)} \quad -(6)$$

et cyclicum

i.e. by a cyclic relation between spin connection vectors:

$$\boxed{\underline{\nabla} \times \underline{\omega}^{(1)*} = i\, \underline{\omega}^{(2)} \times \underline{\omega}^{(3)} \quad \text{et cyclicum}} \quad -(7)$$

Under this condition there is a $\underline{W}^a$ potential

) 3d) no magnetic field.

From Cartan geometry the magnetic field is also defined by:

$$\underline{B}^a = \underline{\nabla} \times \underline{W}^a - \underline{\omega}^a{}_b \times \underline{W}^b \quad -(8)$$

This equation must be used in the complex circular basis, where cross products must be defined as in Eq. (5). For example, the index of the $\underline{B}^{(3)}$ field is:

$$a = (3) \quad -(9)$$

So:

$$\underline{B}^{(3)*} = \underline{B}^{(3)} = \underline{\nabla} \times \underline{W}^{(3)*} - i\underline{\omega}^a{}_{(2)} \times \underline{W}^{(2)}$$

$$= \underline{\nabla} \times \underline{W}^{(3)*} - i\,\underline{\omega}^{(1)} \times \underline{W}^{(2)} \quad -(10)$$

so

$$\boxed{\underline{\omega}^{(3)*}{}_{(2)} = \underline{\omega}^{(1)}} \quad -(11)$$

In general there is summation over repeated b indices in eq. (8), and in the complex circular basis:

$$\underline{B}^{(3)*} = \underline{\nabla} \times \underline{W}^{(3)*} - i\underline{\omega}^{(3)*}{}_{(1)} \times \underline{W}^{(1)} \quad -(12)$$
$$- i\underline{\omega}^{(3)*}{}_{(2)} \times \underline{W}^{(2)} - i\underline{\omega}^{(3)*}{}_{(3)} \times \underline{W}^{(3)}$$

but the basis (5) implies that the cross products $-i\underline{\omega}^{(3)*}{}_{(1)} \times \underline{W}^{(1)}$ and $-i\underline{\omega}^{(3)*}{}_{(3)} \times \underline{W}^{(3)}$ cannot give a vector in the direction of $\underline{B}^{(3)*}$. This implies the result (11).

) Similarly:

$$\underline{\omega}^{(2)} = \underline{\omega}^{(1)*}{}_{(3)} \quad -(13)$$

$$\underline{\omega}^{(1)} = \underline{\omega}^{(3)*}{}_{(2)} \quad -(14)$$

This implies:

$$\underline{\omega}^{(3)*}{}_{(2)} = \epsilon^{(3)*}{}_{(1)(2)}\,\underline{\omega}^{(1)} \quad -(15)$$

$$\underline{\omega}^{(1)*}{}_{(3)} = \epsilon^{(1)*}{}_{(2)(3)}\,\underline{\omega}^{(2)} \quad -(16)$$

$$\underline{\omega}^{(2)*}{}_{(1)} = \epsilon^{(2)*}{}_{(3)(1)}\,\underline{\omega}^{(3)} \quad -(17)$$

where $\epsilon^{(a)*}{}_{(b)(c)}$ is a totally antisymmetric unit tensor in three dimensions and in the complex circular basis. Under these conditions eq. (8) becomes:

$$\underline{B}^{(1)*} = \underline{\nabla} \times \underline{W}^{(1)*} - i\,\underline{\omega}^{(2)} \times \underline{W}^{(3)} \quad -(18)$$

et cyclicum, which is consistent with:

$$\underline{W}^{a} = W^{(0)}\,\underline{\omega}^{a} \quad -(19)$$

Q.E.D.

In general:

$$\underline{W}^{(a)} = W^{(0)}\,\underline{\omega}^{(a)}, \quad -(20)$$

$$a = 1, 2, 3$$

$$\underline{W}^{(1)} \times \underline{W}^{(2)} = i W^{(0)}\,\underline{W}^{(3)*} \quad -(21)$$

which is the W cyclic theorem.

4) If it is assumed that $\underline{W}^{(1)}$ and $\underline{W}^{(2)}$ are plane waves:

$$\underline{W}^{(1)} = \underline{W}^{(2)*} = \frac{W^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi} \quad -(20)$$

then:

$$\underline{W}^{(3)} = W^{(0)}\underline{k} \quad -(21)$$

and

$$\underline{\nabla} \times \underline{W}^{(3)} = \underline{0} \quad -(22)$$

so:

$$\underline{B}^{(3)} = \underline{B}^{(3)*} = \frac{-i}{W^{(0)}}\underline{W}^{(1)} \times \underline{W}^{(2)} \quad -(23)$$

This assumes that the spin connection vectors are plane waves:

$$\underline{\omega}^{(1)} = \underline{\omega}^{(2)*} = \frac{\omega^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi} \quad -(24)$$