340(3). Lamb Shift for the Hydrogenic Wavefunctions

The Lamb shift from the previous note is given by:

$$E_{LS} = -\frac{e^2}{4\pi\epsilon_0 g m^2 c^2}\left\langle \frac{\underline{S}\cdot\underline{L}_1}{r^3}\right\rangle \qquad -(1)$$

$$= -\frac{e^2}{4\pi\epsilon_0 g m^2 c^2}\int \psi^* \frac{\underline{S}\cdot\underline{L}_1}{r^3}\psi \, d\tau$$

If

$$\underline{S}\cdot\underline{L}_1 = S_Z L_{1Z} \qquad -(2)$$

then

$$E_{LS} = -\frac{e^2}{4\pi\epsilon_0 g m^2 c^2}\int \psi^* \frac{S_Z L_{1Z}}{r^3}\psi \, d\tau \qquad -(3)$$

in which

$$\underline{L}_1 = \underline{r}\times\underline{p}_1 \qquad -(4)$$

Using:

$$\underline{S}\cdot\underline{r}\times\underline{p}_1 = \underline{p}_1\cdot(\underline{S}\times\underline{r}) \qquad -(5)$$

Eq. (1) can be expressed as:

$$E_{LS} = -\frac{e^2}{4\pi\epsilon_0 g m^2 c^2}\,\underline{p}_1\cdot\int \psi^* \underline{S}\times\frac{\underline{r}}{r^3}\psi \, d\tau \qquad -(6)$$

in which

$$S_Z\psi = \hbar m_s \psi \qquad -(7)$$

From eqs. (3) and (7):

2)

$$E_{LS} = -\frac{\hbar e^2 m_s}{4\pi\epsilon_0 g m^2 c^2} \int \psi^* \frac{L_{1Z}}{r^3} \psi \, d\tau \quad -(8)$$

In spherical polar coordinates:

$$L_{1Z} = m(vac)\, r^2 \dot{\phi} \sin^2\theta \quad -(9)$$

where $m(vac)$ is the mass of the vacuum particle.

Therefore:

$$E_{LS} = -\frac{\hbar e^2 m_s m(vac)}{4\pi\epsilon_0 g m^2 c^2} \int \psi^* \frac{\dot{\phi} \sin^2\theta}{r} \psi \, d\tau \quad -(10)$$

If it is assumed that in eq. (8):

$$L_{1Z}\psi = m_L \hbar \psi \quad -(11)$$

then:

$$E_{LS} = -\frac{\hbar^2 e^2 m_s m_L}{4\pi\epsilon_0 g m^2 c^2} \int \psi^* \frac{1}{r^3} \psi \, d\tau \quad -(12)$$

where

$$m_s = -S, \ldots, S \quad -(13)$$

and

$$m_L = -L, \ldots, L \quad -(14)$$