40(4): Fluctuating Vacuum Potential and the Lamb Shift.

As in UFT 170 the distance between the electron and the nucleus is assumed to fluctuate by $\delta \underline{r}$, so the Coulomb potential fluctuates by:

$$\Delta U = U(\underline{r} + \delta \underline{r}) - U(\underline{r}) \quad -(1)$$

where

$$U_0 = \frac{-e^2}{4\pi\epsilon_0 r} \quad -(2)$$

Using a Maclaurin series expansion it can be shown that:

$$\langle \Delta U \rangle = \frac{1}{6} \langle (\delta \underline{r})^2 \rangle \nabla^2 U_0 \quad -(3)$$

where

$$\langle \nabla^2 U_0 \rangle = \left\langle \nabla^2 \left(\frac{-e^2}{4\pi r \epsilon_0} \right) \right\rangle \quad -(4).$$

$$= \frac{e^2}{\epsilon_0} |\psi(0)|^2 =$$

This expectation value vanishes for the $2P_{1/2}$ state of the hydrogen atom, but for the $2S_{1/2}$ state it is given by:

$$|\psi_{2S}(0)|^2 = \frac{1}{8\pi a_0^3} \quad -(5)$$

where a_0 is the Bohr radius:

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m e^2} \quad -(6)$$

So:

$$\langle \Delta U \rangle = \frac{1}{6} \langle (\delta \underline{r})^2 \rangle \frac{e^2}{8\pi\epsilon_0 a_0^3} \quad -(7)$$

because

$$\left\langle \nabla^2 \left(\frac{-e^2}{4\pi\epsilon_0 r} \right) \right\rangle = \frac{e^2}{8\pi\epsilon_0 a_0^3} \quad -(8)$$

) In the usual theory of the Lamb shift the fluctuation $\delta \underline{r}$ is worked out from a fluctuating vacuum electric field $\underline{E}_{vac}$, using a balance of the Newton and Lorentz force laws:

$$\underline{F} = m \frac{d^2}{dt^2} (\delta \underline{r}) = -e \underline{E}_{vac} \quad -(9)$$

However the ECE2 vacuum is defined by the Cartan geometry:

$$T = d \wedge q + \omega \wedge q = 0 \quad -(10)$$

and

$$R = d \wedge \omega + \omega \wedge \omega = 0 \quad -(11)$$

i.e. it is a region in which there exists a tetrad q and spin connection ω, but no torsion T or curvature R. This means that there are no electric or magnetic fields, but there are potentials, and the ECE2 vacuum defines the Aharonov Bohm effects.

Therefore the force $\underline{F}$ in eq. (9) must be defined in terms of the ECE2 vacuum potentials, and the fluctuation $\delta \underline{r}$ also calculated from the vacuum potentials. Therefore the vacuum potentials must be assumed to fluctuate. The ECE2 vacuum potential is:

$$\boxed{U(vac) = e \phi_W = \hbar c \Omega^0 = \hbar \omega(vac)} \quad -(12)$$

so the Coulomb potential between electron and proton in the H atom:

$$U_c = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(13)$$

3) is changed to:

$$U_c \rightarrow U_c + U(vac) \quad -(14)$$

$$= -\frac{e^2}{4\pi\epsilon_0 r} + \hbar\omega(vac)$$

$$= \frac{-e^2}{4\pi\epsilon_0 (r - \delta r)}$$

and $$\Delta U = U(\underline{r} - \delta \underline{r}) - U(r) \quad -(15)$$

as in eq. (1), Q.E.D.

From eq. (14):

$$U(vac) = \hbar\omega(vac) = -\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r - \delta r} - \frac{1}{r}\right) \quad -(16)$$

So the vacuum energy is defined by eq. (16). The latter can be expressed as:

$$\boxed{U(vac) = \hbar\omega(vac) = \frac{e^2\,\delta r}{4\pi\epsilon_0 (r - \delta r) r}} \quad -(17)$$

From eqs (14) and (15):

$$\boxed{\begin{aligned}\Delta U &= U(vac) \quad -(18)\\ &= U(\underline{r} - \delta\underline{r}) - U(\underline{r})\end{aligned}}$$

If $$\delta r \ll r \quad -(19)$$

4) Eq. (17) can be written as:

$$U(\text{vac}) = \hbar\omega(\text{vac}) = \frac{e^2}{4\pi\epsilon_0}\left(\frac{\delta r}{r^2} + \frac{(\delta r)^2}{r^3}\right) \quad -(20)$$

Averaging:

$$\langle U(\text{vac})\rangle = \hbar\langle\omega(\text{vac})\rangle$$
$$= \frac{e^2}{4\pi\epsilon_0}\left(\frac{\langle\delta r\rangle}{r^2} + \frac{\langle(\delta r)^2\rangle}{r^3}\right) \quad -(21)$$

If the vacuum fluctuations are isotropic, then:

$$\langle U(\text{vac})\rangle = \hbar\langle\omega(\text{vac})\rangle = \frac{e^2}{4\pi\epsilon_0 r^3}\langle(\delta r)^2\rangle \quad -(22)$$

i.e.

$$\langle\omega(\text{vac})\rangle = \frac{e^2}{4\pi\epsilon_0\hbar r^3}\langle(\delta r)^2\rangle \quad -(23)$$
$$= \frac{c\alpha}{r^3}\langle(\delta r)^2\rangle$$

where the fine structure constant is:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} = 0.007297351 \quad -(24)$$

Therefore:

$$\boxed{\langle\omega(\text{vac})\rangle = 0.007297351\,\frac{c}{r^3}\langle(\delta r)^2\rangle} \quad -(25)$$

) where

$$\epsilon_0 = 8.854188 \times 10^{-12} \, J^{-1} C^2 m^{-1} \quad -(26)$$

and

$$c = 2.997925 \times 10^8 \, m s^{-1} \quad -(27)$$

Now assume that the Lamb shift is given by the mean vacuum angular frequency $\langle \omega(vac) \rangle$.

From eqs. (7) and (25) the Lamb shift is:

$$\boxed{\langle \Delta U \rangle = \frac{e^2 r^3}{48 \epsilon_0 a_0^3} \frac{\langle \omega(vac) \rangle}{\alpha c}} \quad -(28)$$

for the $2S_{1/2}$ state of atomic hydrogen. The measured value is

$$\Delta f = 1.058 \times 10^9 \, Hz \quad -(29).$$

$$\sim 0.04 \, cm^{-1}$$

So

$$\langle \Delta U \rangle = 2\pi \Delta f \hbar \quad \text{Joules} \quad -(30)$$

For the 2S orbital of H:

$$\frac{r}{a_0} = 4 \quad -(31)$$

So

$$2\pi \hbar \Delta f = \frac{e^2}{3072 \epsilon_0 \alpha c} \langle \omega(vac) \rangle \quad -(32)$$

The result (32) is in the right direction, because the $2S_{1/2}$ state is higher than the $2P_{1/2}$ state. Therefore:

$$\langle \omega(vac) \rangle = \frac{3072 \epsilon_0 \alpha c \cdot 2\pi \hbar \Delta f}{e^2} \quad -(33)$$

$$= 1.62509 \times 10^{12} \text{ radians per second}$$