331(2): The Fine Structure of the Relativistic
Zeeman Effect

From the previous note this is given by:

$$H_1 = -\frac{e}{2m}\left(1 - \frac{P_0^2}{m^2c^2}\right)^{-1/2} \underline{L_0} \cdot \underline{B} \quad -(1)$$

$$\sim -\frac{e}{2m}\left(1 + \frac{1}{2}\frac{P_0^2}{m^2c^2}\right) \underline{L_0} \cdot \underline{B}$$

if $\qquad P_0 \ll mc \quad -(2)$

$$-(3)$$

Therefore:

$$\langle H_1 \rangle = -\frac{e}{2m}\langle \underline{L_0} \rangle \cdot \underline{B} - \frac{e}{2m}\frac{1}{mc^2}\left\langle \frac{P_0^2}{2m} \underline{L_0} \right\rangle \cdot \underline{B}$$

If $\underline{B}$ is aligned in the Z axis then:

$$\underline{L_0} \cdot \underline{B} = L_{0z} B_z \quad -(4)$$

In quantum angular momentum theory:

$$L_{0z} \phi = \hbar m_L \phi \quad -(5)$$

where $\qquad m_L = L, L-1, \ldots, -L \quad -(6)$

Therefore: $\qquad \langle \underline{L_0} \rangle = \hbar m_L, \quad -(7)$

and:

a)

$$\left\langle \frac{p_0^2}{2m} L_{0z} \right\rangle = \hbar m_L \left\langle \frac{p_0^2}{2m} \right\rangle \quad -(8)$$

In atomic hydrogen:

$$\langle H_0 \rangle = \left\langle \frac{p_0^2}{2m} + U \right\rangle = -\frac{e^4}{32\pi^2 \epsilon_0^2 \hbar^2 c^2 n^2}$$

$$= -\frac{\hbar c}{2n^2} \frac{\alpha}{r_B} \quad -(9)$$

where α is the fine structure constant and r_B the Bohr radius. We have:

$$\langle U \rangle = \left\langle -\frac{e^2}{4\pi \epsilon_0 r} \right\rangle = -\frac{\hbar c}{n^2} \frac{\alpha}{r_B} \quad -(10)$$

and

$$\left\langle \frac{p_0^2}{2m} \right\rangle = \left\langle -\frac{\hbar^2 \nabla^2}{2m} \right\rangle = \frac{\hbar c}{2n^2} \frac{\alpha}{r_B} \quad -(11)$$

$$= -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d\tau$$

It follows that:

$$\boxed{\langle H_1 \rangle = -\frac{e\hbar}{2m} m_L B_z \left(1 - \frac{1}{mc^2} \frac{\hbar c}{2n^2} \frac{\alpha}{r_B} \right)}$$

$$-(12)$$

The order of the effect is given by:

3)

$$\therefore \langle H_1 \rangle = -\frac{e\hbar}{2m} m_L B_z \left(1 - \frac{1}{2} \frac{\lambda_c \alpha}{n^2 r_B}\right) - (13)$$

where
$$\lambda_c = \frac{\hbar}{mc} \quad - (14)$$

is the Compton wavelength of the electron.

We have:
$$\lambda_c = 3.861591 \times 10^{-13} \text{ m} \quad - (15)$$
$$r_B = 5.29177 \times 10^{-11} \text{ m} \quad - (16)$$
$$\alpha = 0.007297351 \quad - (17)$$

so:

$$\boxed{\langle H_1 \rangle = -\frac{e\hbar}{2m} m_L B_z \left(1 - 2.\frac{662567 \times 10^{-5}}{n^2}\right)}$$

$$- (18)$$

is atomic H, where H_0 depends only on the principal quantum number n. In other atoms and molecules there is a richer structure.

There is a <u>small</u> n <u>dependent shift</u> in the <u>non-relativistic Zeeman effect</u>, and this can be measured by high resolution studies of atomic spectra.

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