330(5): Comparison of Magnitude of Terms

Compare the usual spin orbit hamiltonian:

$$\langle H_{so} \rangle = \frac{e^2}{8\pi\epsilon_0 m^2 c^2} \left\langle \frac{S \cdot L}{r^3} \right\rangle \quad -(1)$$

with the new term:

$$\langle H_2 \rangle = \frac{e^4}{32\pi^2\epsilon_0^2 m^3 c^4} \left\langle \frac{S \cdot L}{r^4} \right\rangle \quad -(2)$$

Eq (1) is:

$$\langle H_{so} \rangle = \frac{\lambda_c^2}{2 m a_0} \left\langle \frac{S \cdot L}{r^3} \right\rangle \quad -(3)$$

and eq. (2) is:

$$\langle H_2 \rangle = \frac{1}{2m} \left(\frac{\lambda_c^2}{a_0} \right)^2 \left\langle \frac{S \cdot L}{r^4} \right\rangle \quad -(4)$$

where the Bohr radius is:

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m e^2} = 5.29177 \times 10^{-11} \, m \quad -(5)$$

and the Compton wavelength is:

$$\lambda_c = \frac{\hbar}{mc} = 3.861591 \times 10^{-13} \, m \quad -(6)$$

In eqs. (3) and (4):

2)

$$\langle \underline{S} \cdot \underline{L} \rangle = \frac{\hbar^2}{2}\Big(J(J+1) - L(L+1) - S(S+1)\Big) \quad -(7)$$

where:

$$J = L+S,\ L+S-1,\ \ldots,\ |L-S|. \quad -(8)$$

Therefore:

$$\boxed{\frac{\langle H_2 \rangle}{\langle H_{so} \rangle} = \frac{\lambda_c^2}{a_0} \left\langle \frac{1}{r^4} \right\rangle \Big/ \left\langle \frac{1}{r^3} \right\rangle} \quad -(9)$$

where

$$\left\langle \frac{1}{r^4} \right\rangle = \int \psi^* \frac{1}{r^4} \psi \, d\tau \quad -(10).$$

and

$$\left\langle \frac{1}{r^3} \right\rangle = \int \psi^* \frac{1}{r^3} \psi \, d\tau$$

$$= \left(\frac{Z}{a_0}\right)^3 \cdot \frac{1}{n^3 L \left(L+\frac{1}{2}\right)(L+1)} \quad -(11)$$

In these equations:

$$\frac{\lambda_c^2}{a_0} = 2.8179 \times 10^{-15}\ \mathrm{m} \quad -(12)$$

So:

$$\langle H_2 \rangle = \left(\frac{a_0}{Z}\right)^3 n^3 L \left(L+\frac{1}{2}\right)(L+1) \frac{\lambda_c^2}{a_0} \left\langle \frac{1}{r^4} \right\rangle \langle H_{so} \rangle$$

$$-(13)$$

3) For the H atom:

$$Z = 1, \quad -(14)$$

so

$$\boxed{\frac{\langle H_2 \rangle}{\langle H_{so} \rangle} = \lambda_c^2 a_0^2 n^3 L\left(L+\frac{1}{2}\right)(L+1)\left\langle \frac{1}{r^4} \right\rangle} \quad -(15)$$

in which:

$$\lambda_c^2 a_0^2 \left\langle \frac{1}{r^4} \right\rangle \quad -(16)$$

$$\sim \left(\frac{\lambda_c}{a_0}\right)^2 \sim 10^{-4}$$

Therefore $\langle H_2 \rangle$ gives rise to a new type of hyperfine detail that depends on:

$$\left\langle \frac{1}{r^4} \right\rangle = \int \phi^* \frac{1}{r^4} \phi \, d\tau \quad -(17)$$

$$\sim \frac{1}{a_0^4}$$

This is well within the resolution of contemporary spectrometers.