235(5): Hamiltonian Method of Calculating $(p/L)^2$ and the Special Relativistic Orbit.

Consider the Hamiltonian method based on:

$$H = \gamma m c^2 + U \quad -(1)$$

of special relativity. It follows that:

$$H = (p^2c^2 + m^2c^4)^{1/2} + U \quad -(2)$$

so:

$$H_0 = H - mc^2 = \frac{p^2c^2}{H - U + mc^2} + U \quad -(3)$$

$$= \frac{p^2c^2}{E + mc^2} + U$$

Here:

$$E = \gamma m c^2, \quad p = \gamma m v_0 \quad -(4)$$

where

$$v_0^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(5)$$

So:

$$H_0 = \left(\frac{\gamma^2}{1+\gamma}\right) m v_0^2 + U \quad -(6)$$

$$= \left(\frac{E^2}{mc^2(E + mc^2)}\right) m v_0^2 + U$$

$$\xrightarrow[v_0 \ll c]{} \frac{1}{2} m v_0^2 + U$$

2) In eq. (5):

$$L_0 = mr^2 \frac{d\theta}{dt} \quad -(7)$$

is the non relativistic angular momentum, a constant of motion. Also:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(8)$$

It follows that:

$$H_0 - U = \left(\frac{E^2}{mc^2(E+mc^2)}\right)\frac{L_0^2}{mr^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) - (9)$$

and

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = mr^4\left(\frac{H_0-U}{L_0^2}\right)\left(\frac{mc^2(E+mc^2)}{E^2}\right).$$

$$\xrightarrow{v_0 \ll c} 2mr^4\left(\frac{H_0-U}{L_0^2}\right) \quad -(10)$$

$$= mr^4\left(\frac{p_0}{L_0}\right)^2.$$

This is the same as the equation derived from the metric

$$c^2 d\tau^2 = (c^2 - v^2)dt^2 \quad -(11)$$

Therefore the transition from classical to relativistic theory is given by:

3)

$$T = \frac{1}{2}\frac{p_0^2}{m} \rightarrow \left(\frac{E^2}{mc^2(E+mc^2)}\right)\frac{p_0^2}{m} \quad -(11)$$

i.e.

$$\boxed{p_0^2 \rightarrow 2\left(\frac{E^2}{mc^2(E+mc^2)}\right)p_0^2} \quad -(12)$$

So

$$\left(\frac{p_0}{L_0}\right)^2 \rightarrow 2\left(\frac{E^2}{mc^2(E+mc^2)}\right)\left(\frac{p_0}{L_0}\right)^2 \quad -(13)$$

The relativistic orbit is therefore given by:

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 + r^2 = 2r^4\left(\frac{E^2}{mc^2(E+mc^2)}\right)\left(\frac{p_0}{L_0}\right)^2} \quad -(14)$$

In this equation:

$$p_0^2 = m^2 MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(15)$$

and

$$L_0^2 = m^2 \underline{M}G\,\alpha \quad -(16)$$

so

$$\boxed{\left(\frac{p_0}{L_0}\right)^2 = \frac{1}{\alpha}\left(\frac{2}{r} - \frac{1}{a}\right)} \quad -(17)$$

where α is the half right latitude and a the semi major axis. These two observables are

3) related by: $a = \frac{\alpha}{1-\epsilon^2}$ —(18)

The classical orbit is:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{p_0}{L_0}\right)^2 - r^2$$

$$= \frac{r^4}{\alpha}\left(\frac{2}{r} - \frac{1}{a}\right) - r^2 \quad -(19)$$

$$= \frac{2r^3}{\alpha} - \frac{r^4}{\alpha^2}\left(1-\epsilon^2\right) - r^2$$

It can be checked that eq. (19) is the same as that given by: $r = \frac{\alpha}{1+\epsilon\cos\theta}$, $\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right)$ —(20)

for which:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\epsilon^2 r^4}{\alpha^2}\sin^2\theta = \frac{\epsilon^2 r^4}{\alpha^2}\left(1-\cos^2\theta\right) \quad -(21)$$

$$= \frac{\epsilon^2 r^4}{\alpha^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)$$

$$= \frac{\epsilon^2 r^4}{\alpha^2} - r^2 + \frac{2r^3}{\alpha} - \frac{r^4}{\alpha^2}$$

Eqs. (19) and (21) are the same, Q.E.D.

4)

So the relativistic orbit is given by:

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{2r^4}{\alpha}\left(\frac{E^2}{mc^2(E+mc^2)}\right)\left(\frac{2}{r}-\frac{1}{a}\right) \quad -(22)$$

where

$$E = \gamma mc^2 = \left(1-\frac{v_0^2}{c^2}\right)^{-1/2} mc^2 \quad -(23)$$

with

$$v_0^2 = MG\left(\frac{2}{r}-\frac{1}{a}\right) \quad -(24)$$

Eq. (22) is the exact orbit of special relativity, QED.

From numerical work it is known that it is a precessing ellipse (UFT324 and UFT325). It can be compared with the x theory:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(25)$$

and

$$\frac{dr}{d\theta} = \frac{x\epsilon r^2}{\alpha}\sin(x\theta) \quad -(26)$$

in order to find x of special relativity, and also the general precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(\theta_1(\theta))} \quad -(27)$$