328(3): Comparison of Precessing Orbits from Numerical and Model Solutions in Special Relativity

This can be carried out by calculating the quantity $(p / L)^2$, where $\underline{p}$ is the relativistic linear momentum and where $\underline{L}$ is the relativistic angular momentum of special relativity. This ratio can be computed from the lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + U \quad -(1)$$

as in UFT324 and UFT325.

It can also be calculated from the infinitesimal line element of special relativity:

$$c^2 d\tau^2 = (c^2 - v_0^2) dt^2 \quad -(2)$$

Here v_0 is the observer frame or classical velocity, defined by:

$$v_0^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \quad -(3)$$

in plane polar coordinates (r, θ). The Lorentz factor is defined directly from eq. (2):

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(4)$$

From a lagrangian analysis of eq. (2), the relativistic velocity is defined by:

2)
$$v^2 = \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 \quad -(5)$$
$$= \gamma^2 v_0^2 \, .$$

The relativistic angular momentum from eq. (2) is defined by
$$L^2 = \gamma^2 m^2 r^4 \dot{\theta}^2 = \gamma^2 L_0^2 \quad -(6)$$
Here L is a constant of motion of special relativity.

Using:
$$\frac{dr}{d\theta} = \frac{dr}{d\tau}\frac{d\tau}{d\theta} \quad -(7)$$

and
$$L = mr^2 \frac{d\theta}{d\tau} \quad -(8)$$

gives:
$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\frac{d\theta}{d\tau} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad -(9)$$

so eq. (5) can be expressed as:
$$\left(\frac{dr}{d\theta}\right)^2 = r^2\left(\left(\frac{p}{L}\right)^2 r^2 - 1\right) \quad -(10)$$

and
$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(11)$$

Eqs. (10) and (11) can be used to compare the numerical $(p/L)^2$ and numerical $(dr/d\theta)^2$

with analytical counterparts from various models of the orbit.

Computer Algebra and Computation

1) Compute $(p/L)^2$ from the lagrangian (1) using

$$U = -\frac{mMG}{r} \quad -(12)$$

as the gravitational potential. The force between m and M is:

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(13)$$

2) Using $(p/L)^2$ compute the orbit from eq. (10). This gives $(dr/d\theta)^2$ by direct numerical evaluation of the lagrangian of special relativity.

3) Compare the numerical $(dr/d\theta)^2$ with various models of the orbit:

a) The classical conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(14)$$

and

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(15)$$

so

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\epsilon^2 r^4}{\alpha^2}\left(1-\cos^2\theta\right) \quad -(16)$$

where

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right) \quad -(17)$$

4) b) The precessing conic section of x theory:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \qquad (18)$$

and

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{x^2\epsilon^2 r^4}{\alpha^2}\sin^2(x\theta) \qquad (19)$$

$$= \frac{x^2\epsilon^2 r^4}{\alpha^2}\left(1-\cos^2(x\theta)\right)$$

where

$$\cos x\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right) \qquad (20)$$

c) The general precessing conic section:

$$r = \frac{\alpha}{1+\epsilon\cos(\theta_1(\theta))} \qquad (21)$$

from which $(dr/d\theta)^2$ can be evaluated by computer algebra.

Having found $(dr/d\theta)^2$ from these models, the model $(p/L)^2$ can be calculated, and compared with the $(p/L)^2$ from eq. (11) and (1). The most general form of the lagrangian (1) is eq. (21) where θ_1 is a precessing as a function of θ. This function can be found by comparing model (c) with the results of the lagrangian (1).