328(2): Simple Expression for Orbital Precession in Special Relativity

The accurate theory is given by the equations:

$$F(r) = -\frac{\partial U}{\partial r} = \frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 \quad -(1)$$

and

$$L = \gamma m r^2 \dot{\theta} = \text{constant}. \quad -(2)$$

As shown in UFT324 these equations lead to orbital precession. The existence of this precession can be confirmed by considering the relativistic momentum:

$$p^2 = m^2 \left(\left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right) \quad -(3)$$

$$= \gamma^2 m^2 \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \right)$$

where

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(4)$$

where γ is the Lorentz factor.

The classical momentum is defined by

$$p_0^2 = m^2 \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \right) \quad -(5)$$

where

$$\frac{d\theta}{dt} = \frac{L_0}{m r^2}. \quad -(6)$$

Using:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} \quad -(7)$$

) it follows that:

$$p_0^2 = m^2\left(\frac{d\theta}{dt}\right)^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(8)$$

$$= \frac{L_0^2}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)$$

So eq. (8) is:

$$\frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) = \left(\frac{p_0}{L_0}\right)^2 \quad -(9)$$

This same result is obtained from the infinitesimal line element of special relativity:

$$c^2 d\tau^2 = (c^2 - v_0^2)\, dt^2 \quad -(10)$$

where

$$v_0^2 = \frac{p_0^2}{m^2} = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(11)$$

From eq. (10), eq. (9) is generalized to:

$$\frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) = \left(\frac{p}{L}\right)^2 \quad -(12)$$

where

$$\left(\frac{p}{L}\right)^2 = \left(\frac{p_0}{L_0}\right)^2 \quad -(13)$$

Eq. (12) is the relativistic version of Eq. (9).

Now consider:

$$\frac{dr}{d\theta} = \frac{dr}{d\tau}\frac{d\tau}{d\theta} \quad -(14)$$

in which:

$$L = mr^2 \frac{d\theta}{d\tau} = \text{constant.} \quad -(15)$$

It follows that:

$$\frac{dr}{d\theta} = \frac{dr}{d\tau}\left(\frac{mr^2}{L}\right) = \gamma \frac{dr}{dt}\left(\frac{mr^2}{L}\right) \quad -(16)$$

Therefore if:

$$L \sim L_0, \quad -(17)$$

an excellent approximation for planetary orbits, it follows that:

$$\frac{dr}{d\theta} = \gamma \frac{dr}{dt}\left(\frac{mr^2}{L_0}\right) = \gamma\left(\frac{dr}{d\theta}\right)_0 . \quad -(18)$$

Therefore the relativistic orbit is:

$$\boxed{\frac{dr}{d\theta} = \gamma\left(\frac{dr}{d\theta}\right)_0} \quad -(19)$$

In eq. (19):

$$\left(\frac{dr}{d\theta}\right)_0 = \frac{\epsilon r^2 \sin\theta}{\alpha} \quad -(20)$$

i.e.

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(21)$$

4) If it is assumed that:

$$r = \frac{\alpha}{1+\epsilon\cos\theta_1} \quad -(22)$$

where

$$\theta_1 = \theta_1(\theta) \quad -(23)$$

then:

$$\boxed{\gamma = \frac{d\theta_1}{d\theta}} \quad -(24)$$

If it is assumed that:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(25)$$

then

$$\boxed{x = \gamma} \quad -(26)$$

Therefore special relativity produces a precessing orbit, confirming the results of UFT324.

The Lorentz factor is defined by eq. (4), where v_0 is defined by

$$v_0^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(27)$$

In the solar system the precession is very tiny, a few arc seconds, so to an excellent approximation:

$$v_0^2 = mG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(28)$$

where v_0 is the orbital velocity of the Newtonian orbit.

So in this approximation:

$$\gamma = \left(1 - \frac{mG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)\right)^{-1/2} \quad -(29)$$

$$\xrightarrow[v_0 \ll c]{} 1$$

At the perihelion:

$$r_{min} = a(1-\epsilon) \quad -(30)$$

so

$$\gamma = \left(1 - \frac{mG}{ac^2}\left(\frac{2}{1-\epsilon} - 1\right)\right)^{-1/2}$$

$$= \left(1 - \frac{mG}{ac^2}\left(\frac{1+\epsilon}{1-\epsilon}\right)\right)^{-1/2} \quad -(31)$$

To an excellent approximation:

$$\boxed{\gamma = 1 + \frac{mG}{2ac^2}\left(\frac{1+\epsilon}{1-\epsilon}\right)} \quad -(32)$$

Eq. (32) can be written as:

$$\gamma = 1 + \frac{mG(1+\epsilon)^2}{2ac^2(1-\epsilon^2)} \quad -(33)$$

For a rotation of 2π the perihelion advances by:

$$\Delta\theta = \frac{\pi mG(1+\epsilon)^2}{ac^2(1-\epsilon^2)} \quad -(34)$$

) This compares with a claimed experimental value of

$$\Delta\theta = \frac{6\pi GM}{ac^2(1-\epsilon^2)} \quad -(35)$$

However, the claimed experimental value is based on a number of disputed assumptions, summarized in the Miles Mathis site. It is known that general relativity has been refuted completely, so the results of special relativity are in very good agreement with the order of magnitude of the precession.