328(1): Direct Method of Calculating the Orbit from the Relativistic Lagrangian of Special Relativity

Consider the relativistic lagrangian of special relativity:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U(r) \quad -(1)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(2)$$

and

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad -(3)$$

Eq. (1) corresponds to the infinitesimal line element of special relativity:

$$c^2 d\tau^2 = (c^2 - v^2)\,dt^2 \quad -(4)$$

From eq. (4) it follows that the orbit is given by:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}\right) \quad -(5)$$

where $\underline{p}$ is the relativistic momentum:

$$\underline{p} = \gamma m \underline{v} \quad -(6)$$

and where $\underline{L}$ is the relativistic angular momentum:

$$\underline{L} = \gamma m r^2 \dot{\theta}. \quad -(7)$$

The latter is a constant of motion so:

$$\frac{d\underline{L}}{dt} = \underline{0}. \quad -(8)$$

In UFT324 it was shown that the lagrangian (1)

2) produces:

$$\ddot{r} = \frac{\left(-\gamma^2 v^2 + \gamma^2 \dot{r}^2 - c^2\right) mG + r\left(\gamma^3 v^4 + \gamma c^2 v^2\right) + r\dot{r}^2\left(-\gamma^3 v^2 - \gamma c^2\right)}{r^2\left(\gamma^3 v^2 + \gamma c^2\right)} \quad -(9)$$

and

$$\ddot{\theta} = \frac{\gamma \dot{r} \dot{\theta} mG + r\dot{r}\dot{\theta}\left(-2\gamma^2 v^2 - 2c^2\right)}{r^2\left(\gamma^2 v^2 + c^2\right)} \quad -(10)$$

Eqn. (9) is the relativistic Leibnitz equation of orbits:

$$\ddot{r} = r\dot{\theta}^2 - \frac{mG}{r^2} \quad -(11)$$

and eq. (10) is the relativistic equivalent of:

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \quad -(12)$$

Eqs (11) and (12) are obtained in the limit:

$$\gamma \rightarrow 1, \quad -(13)$$

i.e. the Newtonian limit.

In order to deduce the relativistic orbit, the ratio $(p/L)^2$ must be calculated from eqs. (9) and (10). Here:

$$p^2 = \gamma^2 m^2 v^2 \quad -(14)$$

and

$$L^2 = \gamma^2 m^2 r^4 \dot{\theta}^2 \quad -(15)$$

This procedure can be carried out with:

3)
$$\dot{r} = \int \ddot{r} \, dt \quad -(16)$$

and
$$\dot{\theta} = \int \ddot{\theta} \, dt \quad -(17)$$

The ratio $(p/L)^2$ has been investigated in papers such as UFT 203, UFT 233 and UFT 324. For the x theory for example:

$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^2}\left(1 + r^2\left(\frac{x\epsilon}{\alpha}\right)^2 \sin^2(x\theta)\right) \quad -(18)$$

$$\xrightarrow[x \to 1]{} \left(\frac{p}{L}\right)^2_N$$

where $(p/L)^2_N$ is the Newtonian limit:

$$\left(\frac{p}{L}\right)^2_N = \frac{1}{r^2}\left(1 + \frac{r^2\epsilon^2}{\alpha^2} \sin^2\theta\right) \quad -(19)$$

For eq. (18):
$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(20)$$

and for eq. (19):
$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(21)$$

From eq. (5):
$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(22)$$

The x theory of eq. (21) relies on the

4) assumption:

$$\theta + \Delta\theta = \theta(1 + x) \quad -(23)$$

So it is assumed that:

$$\Delta\theta = x\theta \quad -(24)$$

As shown in UFT 324, eq. (24) does not produce the relativistic orbit from eqs. (9) and (10). The relativistic orbit precesses in a different way from the orbit of x theory. The orbit from eqs. (9) and (10) is also the orbit of the Sommerfeld atom after Sommerfeld quantization.

More generally:

$$r = \frac{\alpha}{1 + \epsilon\cos(\theta + \Delta\theta)} \quad -(25)$$

where

$$\theta = \theta(r) \text{ and } \Delta\theta = \Delta\theta(r) \quad -(26)$$

It follows that:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\frac{\epsilon^2}{\alpha^2}\left(\frac{d\Delta\theta}{d\theta}\right)^2\left(1 - \cos^2(\theta + \Delta\theta)\right) \quad -(27)$$

$$= r^4\frac{\epsilon^2}{\alpha^2}\left(\frac{d\Delta\theta}{d\theta}\right)^2\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)$$

In the x theory:

$$x = \frac{d\Delta\theta}{d\theta} \quad \text{---} (28)$$

The true orbit may be found from eqs. (9) and (10), if it is assumed that the orbit is governed by special relativity, so $d\Delta\theta / d\theta$ may be calculated. The method depends on integrating eqs. (16) and (17) numerically to give $(p/L)^2$.