327(9): Summary of the Results of the Minkowski Metric Method

The Minkowski metric is represented by the infinitesimal line element:

$$c^2 d\tau^2 = c^2 dt^2 - v_0^2 dt^2 \quad -(1)$$

where

$$v_0^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(2)$$

Therefore:

$$mc^2 = mc^2\left(\frac{dt}{d\tau}\right)^2 - m\left(\frac{dr}{d\tau}\right)^2 - mr^2\left(\frac{d\theta}{d\tau}\right)^2 \quad -(3)$$

From eq (1) the Lorentz factor is

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(4)$$

The relativistic total energy is:

$$E = \gamma mc^2 \quad -(5)$$

and the relativistic angular momentum is

$$L = \gamma mr^2 \frac{d\theta}{dt} = mr^2 \frac{d\theta}{d\tau} \quad -(6)$$

The relativistic linear momentum is:

$$\underline{p} = \gamma m \underline{v}_0 \quad -(7)$$

so

$$p^2 = \gamma^2 m^2 v_0^2 \quad -(8)$$

It follows that:

$$\frac{p^2}{m} = m\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(9)$$

2) Therefore eq. (3) is:

$$E^2 = p^2c^2 + m^2c^4 \quad -(10)$$

The orbit is given by:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{E^2 - m^2c^4}{c^2L^2} - \frac{1}{r^2}\right)$$

$$= r^4\left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}\right) \quad -(11)$$

where

$$\frac{p^2}{L^2} = \frac{p_0^2}{L_0^2} \quad -(12)$$

and where:

$$p_0 = mv_0 \; ; \quad L_0 = mr^2\frac{d\theta}{dt} \quad -(13)$$

are the classical momentum and angular momentum respectively.

Therefore the orbit can be given a relativistic interpretation (11) and a Newtonian interpretation:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{p_0^2}{L_0^2} - \frac{1}{r^2}\right) \quad -(14)$$

In the Newtonian interpretation the momentum p_0 is defined by the hamiltonian:

3)

$$H = \frac{p_0^2}{2m} + U \quad -(15)$$

So

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{L_0^2}\left(2m\left((H-U) - \frac{L_0^2}{2mr^2}\right)\right) \quad -(16)$$

and

$$r = \frac{\alpha}{1+\epsilon\cos\theta}, \quad -(17)$$

where α is the half right latitude and ϵ the eccentricity. Therefore in the Newtonian interpretation the orbit does not precess.

From eq. (17):

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\epsilon^2}{\alpha^2} r^4 \sin^2\theta = r^4\left(\frac{p_0^2}{L_0^2} - \frac{1}{r^2}\right) \quad -(18)$$

where

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(19)$$

It follows that:

$$\left(\frac{p_0}{L_0}\right)^2 = \frac{1}{\alpha}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(20)$$

where the semi major axis is:

$$a = \frac{\alpha}{1-\epsilon^2}. \quad -(21)$$

The Newtonian orbital velocity is:

$$v_0^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(22)$$

and

$$L_0^2 = m^2 MG\alpha \quad -(23)$$

so it follows self consistently that:

$$\frac{p_0^2}{L_0^2} = \frac{m^2 v_0^2}{L_0^2} \quad -(24)$$

i.e

$$p_0 = m v_0 \quad -(25)$$

Q.E.D. The Newtonian potential energy is

$$U = -mMG/r. \quad -(26)$$

The infinitesimal line element (1) is derived from the Lorentz transform invariant:

$$x^\mu x_\mu = x^{\mu\prime} x_\mu{}^\prime, \quad -(27)$$

and contains no reference to a force or potential. It gives the key result that the orbit can be interpreted relativistically as in eq. (11), or classically as in eq. (14).

In the relativistic interpretation, eq. (11), the relativistic angular momentum L is a constant of motion, so:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4 \left(\frac{\gamma^2 p_0^2}{L^2} - \frac{1}{r^2}\right) \quad -(28)$$

In this equation:

$$\gamma^2 = \left(1 - \frac{p_0^2}{m^2 c^2}\right)^{-1} \quad -(29)$$

where

$$p_0 = m v_0 \quad -(30)$$

5) The orbit may also be written as:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{E^2 - m^2c^4}{c^2L^2}\right) - \frac{1}{r^2} \quad -(31)$$

The nature of the orbit depends on the interpretation given to the term $E^2 - m^2c^4$. The Newtonian interpretation is the limit of the relativistic kinetic energy:

$$T = (\gamma - 1)mc^2 \xrightarrow[v \ll c]{} \frac{1}{2}mv^2 \quad -(32)$$

where

$$T = E - mc^2. \quad -(33)$$

As shown in UFT 324 and UFT325, the relativistic hamiltonian

$$H = \gamma mc^2 + U \quad -(34)$$

and the relativistic lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(35)$$

give a precessing orbit. This was demonstrated numerically using a scatter plot method. The metric corresponding to eqs. (34) and (35) is eq. (1), whose orbit is eq. (31). Therefore eq. (31) must give a precessing orbit in order to be consistent with eqs. (34) and (35). In the Newtonian limit eqs. (34) and (35) reduce to:

$$H_0 = \frac{p_0^2}{2m} + U \quad -(36)$$

$$\mathcal{L}_0 = \frac{p_0^2}{2m} - U \quad -(37)$$

6) So in the Newtonian limit eq. (31) gives the conic section (17). The relativistic hamiltonian (34) is:

$$H = \left(c^2 p^2 + m^2 c^4\right)^{1/2} + U \quad - (38)$$

So p can be defined from eq. (38):

$$c^2 p^2 + m^2 c^4 = (H - U)^2. \quad - (39)$$

Various approximations for p may be developed from eq. (39) by factorizing:

$$(H - U)^2 - m^2 c^4 = c^2 p^2 \quad - (40)$$

$$= \left(H - U - mc^2\right)\left(H - U + mc^2\right)$$

as in previous work. Therefore:

$$H - U - mc^2 = \frac{c^2 p^2}{H - U + mc^2}, \quad - (41)$$

and

$$H - mc^2 = \frac{c^2 p^2}{H - U + mc^2} + U \quad - (42)$$

Eq. (42) resembles the classical hamiltonian:

$$H_0 = \frac{p_0^2}{2m} + U \quad - (43)$$

and eq. (42) must reduce to eq. (43) in the limit:

$$v \ll c. \quad - (44)$$

In the Dirac type approximation:

$$U \ll H \sim mc^2 \quad - (45)$$

7) This is a rough approximation that is accepted because it reproduces experimental data satisfactorily, for example atomic spectra. The hamiltonian H_0 is defined as:

$$H_0 = H - mc^2 \quad -(46)$$

Using eqs. (45) and (46):

$$H_0 = \frac{c^2 p^2}{2mc^2 - U} + U \quad -(47)$$

$$= \frac{p^2}{2m\left(1 - \frac{U}{2mc^2}\right)} + U,$$

$$\boxed{H_0 \sim \frac{p^2}{2m}\left(1 + \frac{U}{2mc^2}\right) + U} \quad -(48)$$

The factor 2 in the brackets of the RHS of eq. (48) is the Thomas factor.

Therefore in the Dirac approximation, from a comparison of eqns. (43) and (48):

$$p_0^2 \rightarrow p^2\left(1 + \frac{U}{2mc^2}\right) \quad -(49)$$

where

$$U = -\frac{mMG}{r}. \quad -(50)$$

Therefore

$$\boxed{p_0^2 \rightarrow p^2\left(1 - \frac{MG}{2c^2 r}\right)} \quad -(51)$$

8) Conversely, from eq. (47):

$$p^2 = \left(1 - \frac{U}{2mc^2}\right) p_0^2 \quad -(52)$$

$$= \left(1 + \frac{MG}{2c^2 r}\right) p_0^2$$

In the Dirac approximation therefore, from eqns. (11) and (52):

$$\left(\frac{dr}{d\theta}\right)^2 = r^4 \left(\frac{1}{L^2}\left(1 + \frac{MG}{2c^2 r}\right) p_0^2 + \frac{1}{r^2}\right) \quad -(53)$$

where L is a constant of motion:

$$\frac{dL}{dt} = 0 \quad -(54)$$

Eq. (53) introduces a small correction to the Newtonian orbit so to an excellent approximation:

$$p_0^2 = 2m(H_0 - U) \quad -(55)$$

$$= 2m\left(H_0 + \frac{mMG}{r}\right)$$

Therefore:

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)_N^2 + \left(\frac{dr}{d\theta}\right)_1^2 \quad -(56)$$

9)

It is assumed that this function (56) produces the precessing orbit:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(57)$$

$$-(58)$$

so:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{x^2\epsilon^2}{\alpha^2} r^4 \sin^2 x\theta = r^4\left(\left(\frac{p_0^2}{L^2}+\frac{1}{r^2}\right)+\left(\frac{mG}{2c^2r}\right)\frac{p_0^2}{L^2}\right)$$

in which

$$\sin^2(x\theta) = 1-\cos^2(x\theta) \quad -(59)$$

From eq. (57):

$$\cos^2(x\theta) = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2 \quad -(60)$$

so

$$x^2 = \frac{\frac{\alpha^2}{\epsilon^2}\left(\frac{p_0^2}{L^2}+\frac{1}{r^2}+\left(\frac{mG}{2c^2r}\right)\frac{p_0^2}{L^2}\right)}{1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2} \quad -(61)$$

$$-(62)$$

For a non-precessing ellipse:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{p_0^2}{L^2}+\frac{1}{r^2}\right) = \frac{\epsilon^2 r^4 \sin^2\theta}{\alpha^2}$$

so to an excellent approximation:

$$\boxed{x^2 = 1 + \frac{\frac{\alpha^2}{\epsilon^2}\left(\frac{mG}{2c^2r}\right)\frac{p_0^2}{L^2}}{1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2}} \quad -(63)$$