327(4): Cardano Solution of a Cubic

We wish to find the roots of the cubic:

$$\frac{2mH}{L^2} - x^2 + \frac{2x}{d} + \frac{2L^2}{dm^2c^2}x^3 = 0 \quad -(1)$$

$$:= A - x^2 + Bx + Cx^3$$

This can be expressed as:

$$x^3 - \frac{1}{C}x^2 + \frac{B}{C}x + \frac{A}{C} = 0 \quad -(2)$$

All cubics have one positive root or three positive roots. We wish to factorize eq. (2), so:

$$(x - x_1)(x - x_2)(x - x_3) = 0 \quad -(3)$$

with roots: $x = x_1, \; x = x_2, \; x = x_3$. $-(4)$

Eq. (2) is:

$$ax^3 + bx^2 + cx + d = 0 \quad -(5)$$

where: $a = 1, \; b = -\frac{1}{C}, \; c = \frac{B}{C}, \; d = \frac{A}{C}$ $-(6)$

The Cardano solution of eq. (5) is:

$$x = \left(q + \left(q^2 + (r - p^2)^3\right)^{1/2}\right)^{1/3} + \left(q - \left(q^2 + (r - p^2)^3\right)^{1/2}\right)^{1/3} + p \quad -(7)$$

where: $p = \frac{-b}{3a}, \; q = p^3 + \frac{(bc - 3ad)}{6a^2}, \; r = \frac{c}{3a}$ $-(8)$

2) The three positive roots of eq. (2) are found from eqs. (6) to (8). Therefore:

$$x^3 - \frac{1}{C_1}x^2 + \frac{B}{C_1}x + \frac{A}{C_1} \qquad -(9)$$

$$= (x - x_1)(x - x_2)(x - x_3) = 0$$

Finally the integral to be evaluated is:

$$\theta = \int \frac{dx}{(A - x^2 + Bx + C_1 x^3)^{1/2}} \qquad -(10)$$

$$\theta = C_1^{1/2} \int \frac{dx}{\left(x^3 + \frac{B}{C_1}x - \frac{x^2}{C_1} + \frac{A}{C_1}\right)^{1/2}}$$

where:

$$C_1 = \frac{2L^2}{\alpha m^2 c^2}; \quad B = \frac{2}{\alpha}, \qquad -(11)$$

$$A = \frac{2mH}{L^2}, \quad \alpha = \frac{L^2}{m^2 M G}.$$

$$-(12)$$

The integral:

$$I = \int \frac{dx}{((x - x_1)(x - x_2)(x - x_3))^{1/2}}$$

$$= \frac{2(x - x_1)^{3/2}\left(\frac{x - x_2}{x - x_1}\right)^{1/2} F\left(\sin^{-1}\left(\frac{x_2 - x_1}{x - x_1}\right)^{1/2} \middle| \left(\frac{x_1 - x_3}{x_1 - x_2}\right)\right)}{(x_2 - x_1)^{1/2}((x - x_1)(x - x_2)(x - x_3))^{1/2}}$$

3) where $x_1 = \frac{1}{r_1}$, $x_2 = \frac{1}{r_2}$, $x_3 = \frac{1}{r_3}$ — (13)

and where $F(x|m)$ is the elliptic integral of the first kind. So — (14)

$$\boxed{\theta = \left(\frac{2L^2}{dm^2c^2}\right) \int \frac{dx}{((x-x_1)(x-x_2)(x-x_3))^{1/2}}}$$

i.e.

$$\theta = \frac{2mG}{c^2} \int \frac{dx}{((x-x_1)(x-x_2)(x-x_3))^{1/2}} \quad \text{— (15).}$$

The result is:

$$\theta = -\frac{4mG}{c^2}\left[\frac{(x-x_1)^{3/2}\left(\frac{x-x_2}{x-x_1}\right)^{1/2} F\left(\sin^{-1}\left(\frac{x_2-x_1}{x-x_1}\right)^{1/2} \Bigg| \left(\frac{x_1-x_3}{x_1-x_2}\right)\right)}{(x_2-x_1)^{1/2}((x-x_1)(x-x_2)(x-x_3))^{1/2}}\right]$$

— (16)