327(6) : Method for Calculation of Perihelion Precession

In view of the various errors uncovered recently in the calculation of the Einsteinian perihelion precession this note suggests new ways of calculation. Firstly compare the Newtonian and Einsteinian hamiltonians, respectively:

$$H(\text{Newton}) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + U \quad -(1)$$

and

$$H(\text{Einstein}) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \left(1 - \frac{r_0}{r} \right) \right) + U \quad -(2)$$

where

$$r_0 = \frac{2MG}{c^2} \quad -(3)$$

and

$$L_0 = m r^2 \dot{\theta} \quad -(4)$$

is the classical angular momentum.

It follows that: $\quad -(5)$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \left(\frac{2}{m}(H - U) - \frac{L_0^2}{2mr^2}\left(1 - \frac{r_0}{r}\right) \right)^{1/2}$$

where

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2}. \quad -(6)$$

Therefore: $\quad -(7)$

$$\frac{d\theta}{dr} = \frac{L_0}{r^2} \cdot \frac{1}{\left(2m\left(H - U - \frac{L_0^2}{2mr^2}\left(1 - \frac{r_0}{r}\right)\right)\right)^{1/2}}$$

where

$$U = -\frac{mMG}{r} = -\frac{k}{r} \quad -(8)$$

2)

Now make the change of variable:

$$u = \frac{1}{r}, \quad \frac{du}{dr} = -\frac{1}{r^2}, \quad dr = -r^2 du \quad -(8)$$

so

$$\theta(u) = -\int \frac{L_0\, du}{\left(2m\left(H - U - u^2 \frac{L_0^2}{2m}(1 - ur_0)\right)\right)^{1/2}} \quad -(9)$$

$$= -\int \frac{L_0\, du}{\left(2m\left(H + ku - \frac{L_0^2}{2m}u^2 + \frac{L_0^2}{2m}r_0 u^3\right)\right)^{1/2}}$$

The Newtonian orbit is defined by:

$$\theta_{Newton}(u) = -\int \frac{L_0\, du}{\left(2m\left(H + ku - \frac{L_0^2}{2m}u^2\right)\right)^{1/2}} \quad -(10)$$

It is known that eq. (10) produces:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta}, \quad -(11)$$

the conic section, and it is known that the theory is such that

$$r_0 << r \quad -(12)$$

for planetary orbits.

Eq. (9) can be written as: $\quad -(13)$

$$\theta(u) = -\int \frac{L_0\, du}{\left(2m\left(H + ku - \frac{L_0^2}{2m}u^2\right)\left(1 + \frac{Au^3}{2m\left(H + ku - \frac{L_0^2}{2m}u^2\right)}\right)\right)^{1/2}}$$

3) where

$$A := \frac{r_0 L_0^2}{2m} \quad -(14)$$

For planetary orbits:

$$Au^3 << 2m\left(H + ku - \frac{L_0^2}{2m}u^2\right) \quad -(15)$$

So:

$$\left(1 + \frac{Au^3}{\left(2m\left(H + ku - \frac{u^2 L_0^2}{2m}\right)\right)}\right)^{-1/2} \quad -(16)$$

$$\sim 1 - \frac{Au^3}{4m\left(H + ku - \frac{u^2 L_0^2}{2m}\right)}$$

$$-(17)$$

and:

$$\theta(u) = -\int \frac{L_0\, du}{\left(2m\left(H + ku - \frac{L_0^2}{2m}u^2\right)\right)^{1/2}} + A\int \frac{L_0 u^3\, du}{\left(2m\left(H + ku - \frac{L_0^2}{2m}u^2\right)\right)^{3/2}}$$

Therefore another type of orbit is added to the conic section, which comes from the first integral. The second integral in eq. (17) is of the type:

$$I = \int \frac{u^3\, du}{(a + bu - cu^2)^{3/2}} \quad -(18)$$

The Wolfram integrator gives an analytical result for eq. (18):

4) The final result is:

$$\theta = \cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right)\right) + f(r) \quad -(19)$$

where
$$f(r) = A \int \frac{L_0 u^3 \, du}{\left(2m\left(H + ku - \frac{L_0^2}{2m} u^2\right)\right)^{3/2}} \quad -(20)$$

The angle of precession is defined by:

$$\boxed{\Delta\theta = f(r)} \quad -(21)$$

To an excellent approximation, the Newtonian values can be used for the two constants of motion, H and L_0. The hamiltonian is defined by:

$$H = \frac{1}{2} m v^2 + U \quad -(22)$$

The Newtonian orbital velocity is:

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(23)$$

where a is the semi major axis of the ellipse. Therefore

$$H = -\frac{mG}{2a} \quad -(24)$$

The angular momentum is defined by:

$$L_0^2 = m^2 MG\alpha \quad -(25)$$

5) where α is the half right latitude.

Measurements can be taken at the perihelion,

where:
$$a = \frac{\alpha}{1-\epsilon^2} \quad -(26)$$

and the perihelion is defined by:

$$r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \quad -(27)$$

Therefore the integral (20) can be worked out exactly using these equations, and $\Delta\theta$ measured at the perihelion and calculated.

This is an exact test of the Einstein theory.
