327(3): The Three Roots Needed to Solve the Einstein Integral

The problem discussed by Vankov is the same as that described in Note 325(4). The integral to be worked out is:

$$\theta = \int \left(\frac{2mH}{L^2} - u^2 + \frac{2u}{\alpha}\left(1 + \frac{L^2}{m^2c^2}u^2\right)\right)^{-1/2} du \quad -(1)$$

where $u = \frac{1}{r}$.

It is necessary to factorize the cubic as follows:

$$\frac{2mH}{L^2} - u^2 + \frac{2u}{\alpha}\left(1 + \frac{L^2}{m^2c^2}u^2\right) \quad -(2)$$

$$= \alpha\left(u - u_1\right)\left(u - u_2\right)\left(u - u_3\right),$$

Then: $\frac{1}{\alpha} = u_1 + u_2 + u_3$

$$\theta = \int \frac{du}{(u-u_1)(u-u_2)(u-u_3)} \quad -(3)$$

The solution of eq. (3) is known exactly:

2)

$$\theta = \frac{2(u-u_1)^{3/2}\left(\frac{u-u_2}{u-u_1}\right)^{1/2}\left(\frac{u-u_3}{u-u_1}\right)^{1/2} F\left(\sin^{-1}\left(\frac{u_2-u_1}{u-u_1}\right)^{1/2} \middle| \frac{u_1-u_3}{u_1-u_2}\right)}{(u_2-u_1)^{1/2}\left((u-u_1)(u-u_2)(u-u_3)\right)^{1/2}} \quad -(4)$$

where $F(a|b)$ is the elliptical integral of the first kind. Here:

$$u_1 = \frac{1}{r_1}, \quad u_2 = \frac{1}{r_2}, \quad u_3 = \frac{1}{r_3} \quad -(5)$$

The notation used by Einstein is:

$$\theta = \int_{\alpha_1}^{\alpha_2} \frac{dx}{\left(\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3\right)^{1/2}} \quad -(6)$$

where α_1 and α_2 are roots of:

$$\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 = 0 \quad -(7)$$

However the exact solution given by Vankov is:

$$\theta = \int_{x_1}^{x_2} \frac{dx}{\left(\alpha(x-x_1)(x-x_2)(x-x_3)\right)^{1/2}} \quad -(8)$$

where

$$\frac{1}{\alpha} = x_1 + x_2 + x_3 \quad -(9)$$

3) However, the most direct and simplest method is to use Maxima to find u_1, u_2 and u_3 in terms of $\frac{2mH}{L^2}$, $\frac{2}{\alpha}$ and $\frac{2L^2}{\alpha m^2 c^2}$, where α is defined by $L^2 / (m^2 M G)$. The origin of the integral (1) is the potential of general relativity:

$$U = -\frac{mMG}{r}\left(1 + \frac{L^2}{m^2 c^2 r^2}\right) \quad -(10)$$

It is known from previous work that this does not give an orbit of the type:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(11)$$

However, Vankov shows that the analytical solution obtained by Einstein assumes classical initial conditions and small perturbations of the potential, kinetic and total energies and of the angular momentum. Einstein assumed that the impact of the GR term on the classical roots x_1 and x_2 is negligible. However this leads to

$$r = \frac{r_0}{1 + e\cos(\omega\theta)} \quad -(12)$$

which is eq. (3.7), p. 20 of Vankov. Eqs. (11) and (12) have the same structure, but it is known that they do not result from eq. (10). So the Einstein method is incorrect, QED