327(1): Orbital Theory from Special Relativity

In UFT324 and UFT325 it was shown that perihelion precession can be described with special relativity. In this note that conclusion is confirmed using the method first developed in UFT203 using the infinitesimal line element of special relativity. It is based on the Minkowski metric and infinitesimal line element:

$$c^2 d\tau^2 = (c^2 - v^2)\, dt^2 \quad -(1)$$

where

$$v^2 dt^2 = dr^2 + r^2 d\theta^2 \quad -(2)$$

in plane polar coordinates (r, θ). So:

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(3)$$

and

$$\mathcal{L} = \frac{1}{2} mc^2 = \frac{1}{2} mc^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{d\theta}{d\tau}\right)^2 r^2 \quad -(4)$$

The total relativistic energy is:

$$E = \gamma mc^2 = \left(\frac{dt}{d\tau}\right) mc^2 \quad -(5)$$

and the relativistic angular momentum is:

$$L = \gamma m r^2 \frac{d\theta}{dt} = m r^2 \frac{d\theta}{d\tau} \quad -(6)$$

From eq. (1):

2)

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{c^2}{c^2 - v^2} = \gamma^2 \quad -(7)$$

Fm eq. (4):

$$mc^2 = \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{mr^2} \quad -(8)$$

where:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \left(\frac{d\theta}{d\tau}\right)^2 = \left(\frac{L}{mr^2}\right)^2 \left(\frac{dr}{d\theta}\right)^2 \quad -(9)$$

Therefore:

$$mc^2 = \frac{E^2}{mc^2} - \frac{L^2}{mr^4}\left(\frac{dr}{d\theta}\right)^2 - \frac{L^2}{mr^2} \quad -(10)$$

and

$$\frac{L^2}{mr^2}\left(\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2 + 1\right) = \frac{E^2 - m^2c^4}{mc^2} \quad -(11)$$

so

$$\boxed{\begin{aligned}\left(\frac{dr}{d\theta}\right)^2 &= r^4\left(\frac{E^2 - m^2c^4}{c^2L^2} - \frac{1}{r^2}\right) \quad -(12)\\ &= r^4\left(\frac{p^2}{L^2} - \frac{1}{r^2}\right)\end{aligned}}$$

where:

3)

$$p = \gamma p_0 \quad -(13)$$

and

$$L = \gamma L_0 \quad -(14)$$

where p_0 and L_0 are the non-relativistic momentum and angular momentum respectively.

The classical Newtonian orbit is given by:

$$\left(\frac{dr}{d\theta}\right)^2_N = \left(\frac{2m(H-V)}{L_0^2}\right) r^4 - r^2 \quad -(15)$$

where the classical hamiltonian is given by:

$$H_0 = T + V \quad -(16)$$

and where the classical angular kinetic energy is:

$$T = \frac{p_0^2}{2m} = \frac{1}{2} m v_0^2. \quad -(17)$$

Therefore:

$$p_0^2 = 2m(H_0 - V) \quad -(18)$$

and

$$\boxed{\left(\frac{dr}{d\theta}\right)^2_N = r^4\left(\frac{p_0^2}{L_0^2} - \frac{1}{r^2}\right)} \quad -(19)$$

However:

$$\frac{p^2}{L^2} = \frac{p_0^2}{L_0^2} \quad -(20)$$

4) So the orbit of special relativity is the conic section:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(21)$$

in which the relativistic p and L are used in place of the classical p_0 and L_0.

In the Newtonian theory:

$$\alpha_0 = \frac{L_0^2}{m^2 M G} \quad -(22)$$

For an ellipse:

$$a = \frac{\alpha_0}{1 - \epsilon_0^2} = \frac{mMG}{2|H_0|} \quad -(23)$$

So

$$1 - \epsilon_0^2 = \frac{2\alpha_0 |H_0|}{mMG} \quad -(24)$$

and

$$\epsilon_0^2 = 1 - \frac{2\alpha |H_0|}{mMG} \quad -(25)$$

i.e

$$\epsilon_0^2 = 1 - \frac{2 L_0^2 |H_0|}{m^3 M^2 G^2} \quad -(26)$$

For a hyperbola:

5)

$$\epsilon_0^2 = 1 + \frac{2L_0^2|H_0|}{m^3M^2G^2} \quad -(27)$$

The orbit in special relativity is eq. (21) with the relativistic α and ϵ.

Therefore the classical angular momentum L_0 must be replaced with L, and the classical hamiltonian H_0 by the relativistic hamiltonian:

$$H = \gamma mc^2 + U \quad -(28)$$

So the relativistic α is:

$$\boxed{\alpha = \gamma^2 \frac{L_0^2}{m^2MG} = \gamma^2\alpha_0} \quad -(29)$$

i.e

$$\alpha = \left(1 - \frac{v_0^2}{c^2}\right)^{-1}\alpha_0 \quad -(30)$$

where

$$v_0^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(31)$$

is the Newtonian orbital velocity. Here a is the half major axis of an ellipse. For a

6) hyperbola a is the distance of closest approach:

$$a = R_0 \quad -(32)$$

The perihelion in an ellipse is defined by

$$r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \quad -(33)$$

The distance r_{min} and a can be measured experimentally, so at the perihelion:

$$v_0^2 = MG\left(\frac{2}{r_{min}} - \frac{1}{a}\right) \quad -(34)$$

is known experimentally and so γ is known experimentally at the perihelion.

The eccentricity of the ellipse is also known experimentally and in a first approximation:

$$\epsilon \sim \epsilon_0 \quad -(35)$$

so the relativistic elliptical orbit is:

$$\boxed{r = \frac{\gamma^2 \alpha_0}{1+\epsilon_0 \cos\theta}} \quad -(36)$$

in the approximation (35).

Therefore:

$$\cos\theta = \frac{1}{\epsilon_0}\left(\frac{\gamma^2\alpha_0}{r} - 1\right) \quad -(37)$$

It is known experimentally that the elliptical orbit precesses, so:

$$\cos\theta = \cos(\theta_0 + \Delta\theta) \quad -(38)$$

where θ_0 is the angle transcribed by a non-precessing orbit and where $\Delta\theta$ is the advance over θ_0 in a precessing ellipse.

If $$\theta_0 = 2\pi \quad -(39)$$

then:

$$\cos(2\pi + \Delta\theta) = \cos\Delta\theta \quad -(40)$$

So:

$$\boxed{\cos\Delta\theta = \frac{1}{\epsilon_0}\left(\frac{\gamma^2\alpha_0}{r} - 1\right)} \quad -(41)$$

At the perihelion

$$r = r_{min} \quad -(42)$$

Denote:

$$\theta_0 + \Delta\theta = \theta_0(1 + x) \quad -(43)$$

then:

$$\cos(\theta + x\theta) = \frac{1}{\epsilon_0}\left(\gamma^2 \frac{\alpha_0}{r} - 1\right) \quad -(44)$$

So $$\theta(1+x) = \cos^{-1}\left(\frac{1}{\epsilon_0}\left(\gamma^2\frac{\alpha_0}{r} - 1\right)\right) \quad -(45)$$

Similarly:

$$\theta = 2\pi = \cos^{-1}\left(\frac{1}{\epsilon_0}\left(\frac{\alpha_0}{r} - 1\right)\right) \quad -(46)$$

So:

$$x = \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{\epsilon_0}\left(\gamma^2\frac{\alpha_0}{r} - 1\right)\right) \quad -(47)$$

At ~~the~~ perihelia:

$$\boxed{x = \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{\epsilon_0}\left(\gamma^2\frac{\alpha_0}{r_{min}} - 1\right)\right)} \quad -(48)$$

where $$\gamma^2 = \left(1 - \frac{v_0^2}{c^2}\right)^{-1} \quad -(49)$$

and $$v_0^2 = MG\left(\frac{2}{r_{min}} - \frac{1}{a}\right) \quad -(50)$$