327(5): The Relativistic Ellipticity

It has been shown that the orbit from the Minkowski metric is:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(1)$$

where α and ϵ are the relativistic half right latitude:

$$\alpha = \gamma^2 \alpha_0 \quad -(2)$$

where

$$\alpha_0 = \frac{L_0^2}{m^2 M G} \quad -(3)$$

The relativistic ellipticity is defined by

$$\epsilon^2 = 1 \pm \frac{2 H L^2}{m^3 M^2 G^2} \quad -(4)$$

where H is the relativistic hamiltonian and L the relativistic angular momentum. The hyperbola is defined by the positive sign and the ellipse by the negative sign. The two constants of motion of the relativistic orbit are H and L. These are defined by:

$$H = \gamma m c^2 + U \quad -(5)$$

and

$$L = \gamma L_0 = \gamma m r^2 \dot{\theta} \quad -(6)$$

The non-relativistic hamiltonian is:

$$H_0 = \frac{1}{2} m v_0^2 + U \quad -(7)$$

2) So

$$H = H_0 + \gamma mc^2 - \frac{1}{2}mv_0^2$$

$$= H_0 + \left(1-\frac{v_0^2}{c^2}\right)^{-1/2} mc^2 - \frac{1}{2}mv_0^2 \quad -(8)$$

Now use:

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots \quad -(9)$$

where

$$x = \left(\frac{v_0}{c}\right)^2 \quad -(10)$$

If

$$x \ll 1 \quad -(11)$$

then

$$H \sim H_0 + \frac{3}{8}\frac{v_0^4}{c^4}mc^2$$

$$= H_0 + \frac{3}{8}\frac{mv_0^4}{c^2} \quad -(12)$$

By definition:

$$L^2 = \gamma^2 L_0^2 = \left(1-\frac{v_0^2}{c^2}\right)L_0^2 \quad -(13)$$

Therefore:

$$= \left(1-\frac{v_0^2}{c^2}\right)m^2 MG\alpha_0$$

$$\boxed{\epsilon^2 = 1 \pm 2m^2 MG\alpha_0\left(1-\frac{v_0^2}{c^2}\right)\left(H_0 + \frac{3}{8}\frac{mv_0^4}{c^2}\right)^2}$$

$$-(14)$$

In eq. (14):

$$v_0^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(15)$$

where a is the semimajor axis. Therefore:

$$H_0 = \frac{1}{2}mv_0^2 - \frac{mMG}{r} = -\frac{mMG}{a} \quad -(16)$$

i.e.

$$H_0 = -\frac{MG}{a} \quad -(16)$$

The classical half right latitude is:

$$\alpha_0 = (1-\epsilon_0^2)\,a \quad -(17)$$

The relativistic orbit from the Minkowski metric is therefore:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(18)$$

where

$$\alpha = \left(1 - \frac{v_0^2}{c^2}\right)\alpha_0 \quad -(19)$$

and

$$\epsilon^2 = 1 \pm 2m^2MG\alpha_0\left(1-\frac{v_0^2}{c^2}\right)\left(\frac{3}{8}m\frac{v_0^4}{c^2} - \frac{mMG}{a}\right)^2 \quad -(20)$$

and

$$v_0^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(21)$$

4) <u>Computer Algebra</u>

Using eqns. (18) to (21), r may be found in terms of θ and the orbit plotted. For planetary motion

$$v_0 << c. \quad -(22)$$

The orbit is defined by

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(23)$$

where

$$\theta := (1+x)\,\theta_0 \quad -(24)$$

The classical orbit is defined by:

$$\cos\theta_0 = \frac{1}{\epsilon_0}\left(\frac{\alpha_0}{r} - 1\right) \quad -(25)$$

Therefore:

$$\theta - \theta_0 = x\,\theta_0 \quad -(26)$$

and

$$\frac{\theta}{\theta_0} = 1 + x \quad -(27)$$

If

$$\theta_0 = 2\pi \quad -(28)$$

then

$$\boxed{1 + x = \frac{1}{2\pi}\,\frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right)} \quad -(29)$$