325(1): Integration of the New Binet Equation to Give Orbits from Potentials

The new Binet equation is:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2}(E - U) \quad -(1)$$

where

$$u = \frac{1}{r} \quad -(2)$$

Therefore:

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2m}{L^2}(E - U) - u^2 \quad -(3)$$

The Newtonian potential is:

$$U = -k/r \quad -(4)$$

where

$$k = mMG \quad -(5)$$

and the Newtonian orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(6)$$

Therefore

$$u = \frac{1}{\alpha}(1+\epsilon\cos\theta) \quad -(7)$$

and

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \epsilon^2 + 1 + 2\epsilon\cos\theta \quad -(8)$$

From eqs. (1) and (8):

$$\epsilon^2 + 1 + 2\epsilon\cos\theta = \frac{2m}{L^2}\left(E + \frac{k}{\alpha}(1+\epsilon\cos\theta)\right) \quad -(9)$$

so

$$\epsilon^2 = 1 - \frac{2m\alpha^2 E}{L^2} \quad -(10)$$

$$\alpha = \frac{L^2}{mk}, \quad E = -\frac{k}{\alpha}. \quad -(11)$$

In general the orbit is given by:

$$\frac{d\theta}{du} = \left(\frac{2m}{L^2}(E-U) - u^2 \right)^{-1/2} \quad -(12)$$

So

$$\theta = \int \left(\frac{2m}{L^2}(E-U) - u^2 \right)^{-1/2} du \quad -(13)$$

For the Newtonian orbit:

$$\theta = \int \left(\frac{2m}{L^2}(E+ku) - u^2 \right)^{-1/2} du \quad -(14).$$

and numerical or analytical integration should give the orbit (7).

For the classical precessing orbit:

$$U = -\frac{x^2 L}{m\alpha r} + \frac{1}{2}(x^2-1)\frac{L^2}{mr^2} \quad -(15)$$

So

$$\theta = \int \left(\frac{2m}{L^2}\left(E + \frac{x^2 L}{m\alpha} u\right) - x^2 u^2 \right)^{-1/2} du \quad -(16)$$

and this gives the orbit

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(17)$$

3) These orbits are known so can be used to check a numerical integration procedure.

From previous work the Einsteinian force law is:

$$F(\text{Einstein}) = -\left(\frac{mMG}{r^2} + \frac{3MGL^2}{mc^2r^4}\right) \quad -(18)$$

and does not give a precessing ellipse because the Einsteinian potential is:

$$U = -k\left(u + \frac{L^2}{m^2c^2}u^3\right) \quad -(19)$$

so the Einsteinian orbit is given by:

$$\theta(\text{Einstein}) = \int\left(\frac{2m}{L^2}\left(E + k\left(u + \frac{L^2}{m^2c^2}u^3\right)\right) - u^2\right)^{-1/2} du$$

and this must be integrated numerically. (20)

It is found that:

$$\left(\frac{du}{d\theta}\right)^2(\text{Einstein}) = \left(\frac{du}{d\theta}\right)^2(\text{Newton}) + \frac{2m}{L^2}\frac{kL^2}{m^2c^2}u^3$$

$$= \left(\frac{du}{d\theta}\right)^2(\text{Newton}) + r_0u^3 \quad -(21)$$

) where

$$r_0 = \frac{2MG}{c^2} \quad - (22)$$

This is misattributed to Schwarzschild in the standard model and called the Schwarzschild radius.

However, it is known that:

$$\left(\frac{du}{d\theta}\right)^2 (\text{Newton}) = \frac{\epsilon^2}{\alpha^2} \sin^2\theta \quad - (23)$$

so

$$\boxed{\left(\frac{du}{d\theta}\right)^2_E = \frac{\epsilon^2}{\alpha^2} \sin^2\theta + r_0 u^3} \quad - (24)$$

This differential equation gives the Einstein orbit. It appears to be a new equation, and can be solved for $u = 1/r$ in terms of θ.

When

$$\theta = 2\pi \quad - (25)$$

$$\left(\frac{du}{d\theta}\right)^2_E = \frac{r_0}{r^3} \quad - (26)$$

and

$$\frac{du}{d\theta} = \int \left(r_0 u^3 \right)^{1/2} du \quad - (27)$$

so this is the precession of the Einstein theory.