325(2): Exact Solution of the Einsteinian Orbital Equations.

The two Einsteinian orbital equations are:

$$f_1 = \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2}(E-U) + r_0 u^3 \quad -(1)$$

and
$$f_2 = \frac{1}{\alpha} + \frac{3}{2} r_0 u^2 = \frac{d^2 u}{d\theta^2} + u \quad -(2)$$

Eq. (1) is the integrated Binet equation and eq. (2) is the Binet equation. These must be solved simultaneously. Eq (2) gives:

$$\frac{1}{\alpha}(\text{Einstein}) = \frac{1}{\alpha} + \frac{3}{2}\frac{r_0}{r^2} \quad -(3).$$

Eq. (1) gives:

$$\frac{2m}{L^2}(E-U) \rightarrow \frac{2m}{L^2}(E-U) + \frac{r_0}{r^3} \quad -(4)$$

so
$$T(\text{Einstein}) = T(\text{Newton}) + \frac{L^2}{2m}\frac{r_0}{r^3} \quad -(5)$$

where
$$T = E - U \quad -(6)$$

is the kinetic energy. The Newtonian angular momentum is:

$$L = mr^2\dot{\theta} \quad -(7)$$

so:

2)

$$T(\text{Einstein}) = T(\text{Newton}) + \frac{1}{2} m r r_0 \dot{\theta}^2 \quad -(8)$$

The Einsteinian lagrangian is therefore:

$$\mathcal{L}(E) = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \left(1 + \frac{r_0}{r}\right) - U \quad -(9)$$

and the Einsteinian hamiltonian is:

$$H(E) = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \left(1 + \frac{r_0}{r}\right) + U \quad -(10)$$

Since:

$$r_0 << r \quad -(11)$$

Eq. (10) is to an excellent approximation:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \left(\frac{2}{m}\left(E - U\right) - \frac{L^2}{m^2 r^2}\right)^{1/2} \quad -(12)$$

where

$$L = m r^2 \dot{\theta} \quad -(13)$$

is the constant angular momentum. So:

$$\theta(t) = \int \frac{(1/r^2)\,dr}{\left(2m\left(E + \frac{1}{r} - \frac{L^2}{2mr^2}\right)\right)^{1/2}} \quad -(14)$$

i.e

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(15)$$

where

$$\alpha = \frac{L^2}{m^2 M G}, \quad -(16)$$

and for an ellipse:

3) and for an ellipse:

$$\epsilon = \left(1 - \frac{2EL^2}{m k^2}\right)^{1/2}, \quad -(17)$$

for a hyperbola:

$$\epsilon = \left(1 + \frac{2EL^2}{m k^2}\right)^{1/2} \quad -(18)$$

where

$$k = mMG. \quad -(19)$$

So in this approximation, and using eq. (3):

$$\boxed{\frac{1}{r} = \left(\frac{1}{\alpha} + \frac{3}{2}\frac{r_0}{r^2}\right)\left(1 + \epsilon\cos\theta\right)} \quad -(20)$$

At the perihelion:

$$r = a(1-\epsilon) \quad -(21)$$

so

$$\frac{1}{\alpha}(\text{Einstein}) = \frac{1}{\alpha} + \frac{3mG}{c^2 a^2 (1-\epsilon)^2} \quad -(22)$$

The observed light deflection due to gravitation is

$$\Delta\theta = 2\pi\left(\frac{3mG}{c^2 a(1-\epsilon^2)}\right) \quad -(23)$$

and

$$\frac{1}{\Delta\alpha} = \frac{3}{2}\frac{r_0}{r^2} = \frac{3mG}{c^2 a^2 (1-\epsilon)^2} \quad -(24)$$

4) So: $$\frac{1}{\Delta \alpha} = \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{a}\frac{\Delta\theta}{2\pi} \quad -(25)$$

At the perihelia in eq. (20):

$$\frac{1}{r_{min}} = \left(\frac{1}{\alpha} + \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{a}\frac{\Delta\theta}{2\pi}\right)(1+\epsilon)$$

$$= \frac{1+\epsilon}{\alpha} + \frac{(1+\epsilon)^2}{(1-\epsilon)a}\frac{\Delta\theta}{2\pi}$$

$$= \frac{1}{r_{min}} + \frac{(1+\epsilon)^2}{r_{min}}\frac{\Delta\theta}{2\pi} \quad -(26)$$

So $$\frac{1}{r_{min}} \rightarrow \frac{1}{r_{min}}\left(1 + (1+\epsilon)^2\frac{\Delta\theta}{2\pi}\right) \quad -(27)$$

The overall result is therefore:

$$\frac{1}{r} = \left(\frac{1}{\alpha} + \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{a}\frac{\Delta\theta}{2\pi}\right)(1+\epsilon\cos\theta) \quad -(28)$$

at the perihelia $$r = r_{min} = \frac{\alpha}{1+\epsilon} \quad -(29)$$

So $$\frac{1+\epsilon}{\alpha} \rightarrow \frac{1+\epsilon}{\alpha} + \frac{1}{a}\left(\frac{(1+\epsilon)^2}{1-\epsilon}\frac{\Delta\theta}{2\pi}\right) \quad -(30)$$

i.e. $$\frac{1}{\alpha} \rightarrow \frac{1}{\alpha} + \left(\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{a}\right)\frac{\Delta\theta}{2\pi} \quad -(31)$$

but $$a = \frac{\alpha}{1-\epsilon^2} \quad -(31)$$

so $$\frac{1}{\alpha} \rightarrow \frac{1}{\alpha} + \left(\frac{1-\epsilon^2}{\alpha}\right)\frac{\Delta\theta}{2\pi} \quad -(32)$$

5) However:

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{1}{a} = \frac{(1-\epsilon^2)}{\alpha} = \frac{1}{a} \quad -(33)$$

at the perihelion, so

$$a = \alpha \quad -(34)$$

and

$$\boxed{\frac{1}{\alpha} \rightarrow \frac{1}{\alpha}\left(1+\frac{\Delta\theta}{2\pi}\right)} \quad -(35)$$

at the perihelion, where:

$$\Delta\theta = \frac{6\pi MG}{c^2 a(1-\epsilon^2)} \quad -(36).$$

which is the observed result.