325(8): Expression for x from the Relativistic Lagrangian of ECE2.

The relativistic lagrangian is :

$$\mathcal{L} = -mc^2 f^{1/2} + \frac{mMG}{r} \quad -(1)$$

where

$$\frac{1}{\gamma} = f = 1 - \frac{1}{c^2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right) \quad -(2)$$

and the relativistic hamiltonian is :

$$H = \gamma mc^2 - \frac{mMG}{r} \quad -(3)$$

The Sommerfeld hamiltonian is :

$$H_1 = H - mc^2 = (\gamma - 1)mc^2 - \frac{mMG}{r} \quad -(4)$$

The Euler Lagrange equations are :

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(5)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(6)$$

From eq. (5) :

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \dot{\theta}} = \frac{mc^2}{2} f^{-1/2} \frac{2r^2\dot{\theta}}{c^2}$$

$$= \gamma m r^2 \dot{\theta} \quad -(7)$$

$$:= L$$

and

$$\frac{dL}{dt} = 0 \quad -(8)$$

2) Similarly:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial r}, \quad \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \dot{r}} \quad -(9)$$

Fm eq. (2):

$$\frac{\partial f}{\partial r} = -\frac{2r\dot{\theta}^2}{c^2}, \quad \frac{\partial f}{\partial \dot{r}} = -\frac{2\dot{r}}{c^2} \quad -(10)$$

and

$$\frac{\partial \mathcal{L}}{\partial f} = -\frac{mc^2}{2} f^{-1/2} \quad -(11)$$

Therefore:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{mc^2}{2} f^{-1/2} \cdot \frac{2r\dot{\theta}^2}{c^2} - \frac{\partial U}{\partial r}$$

$$= \gamma m r \dot{\theta}^2 - \frac{\partial U}{\partial r} \quad -(11)$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{mc^2}{2} f^{-1/2} \frac{2\dot{r}}{c^2} = \gamma m \dot{r} \quad -(12)$$

so

$$\gamma m r \dot{\theta}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}\left(\gamma m \dot{r}\right) \quad -(13)$$

i.e.

$$\boxed{\frac{d}{dt}\left(\gamma m \dot{r}\right) - \gamma m r \dot{\theta}^2 = -\frac{\partial U}{\partial r} = F(r) \quad -(14)}$$

and

$$\boxed{L = \gamma m r^2 \dot{\theta}} \quad -(15)$$

3) In the non-relativistic limit:

$$\gamma \rightarrow 1 \quad - (16)$$

and eq. (14) becomes the Leibnitz equation of orbits:

$$m\frac{d^2r}{dt^2} = mr\omega^2 - \frac{mMG}{r^2} \quad - (17)$$

with conserved angular momentum:

$$L = mr^2\omega \quad - (18)$$

where

$$\omega = \dot{\theta} = \frac{d\theta}{dt} \quad - (19)$$

Eq. (14) is therefore the Leibnitz equation in special relativity.

Using the change of variable:

$$\dot{r} = \frac{dr}{dt} = -r^2\frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad - (20)$$

it is found that:

$$\boxed{\begin{aligned}\dot{r} &= -\frac{L}{\gamma m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad - (21) \\ &\text{and} \\ \dot{\theta} &= \frac{L}{\gamma m r^2} \quad - (22)\end{aligned}}$$

As in the previous note it follows that the orbital velocity in special relativity is

4)

$$v^2 = \frac{v_N^2}{1 + \left(\frac{v_N}{c}\right)^2} \quad -(23)$$

where the Newtonian velocity v_N is given by:

$$v_N^2 = \frac{L^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) = MG\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$-(24)$$

where the major semi axis a is defined by:

$$\frac{1}{a} = \frac{1-\epsilon^2}{\alpha}, \quad -(25)$$

where, is the Newtonian orbit:

$$\alpha = r(1 + \epsilon\cos\theta). \quad -(26)$$

So:

$$v_N^2 = \frac{MG}{r}\left(2 - \frac{(1-\epsilon^2)}{1+\epsilon\cos\theta}\right) \quad -(27)$$

$$\xrightarrow[r \to \infty]{} 0$$

The relativistic velocity v is that of a non-Newtonian orbit. The velocity v_N is that of the Newtonian orbit (26). If it is assumed that the non Newtonian orbit is:

5)

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(28)$$

then its velocity in the classical limit is

$$v_c^2 = \left(\frac{L}{m\alpha}\right)^2\left((x^2+1)\frac{\alpha}{r} + x^2(\epsilon^2-1)\right) \quad -(29)$$

$$\xrightarrow[x\to 1]{} MG\left(\frac{2}{r}-\frac{1}{a}\right)$$

using

$$L^2 = m^2 MG\alpha \quad -(30)$$

Its relativistic velocity is therefore: $-(31)$

$$\boxed{v^2 = \frac{v_c^2}{1+\left(\frac{v_c}{c}\right)^2} \xrightarrow[x\to 1]{} \frac{v_N^2}{1+\left(\frac{v_N}{c}\right)^2}}$$

If

$$v_c \ll c \quad -(32)$$

then

$$v_c^2 \sim \frac{v_N^2}{1+\left(\frac{v_N}{c}\right)^2} \quad -(33)$$

i.e.

$$\boxed{\left(\frac{L}{m\alpha}\right)^2\left((1+x^2)\frac{\alpha}{r} + x^2(\epsilon^2-1)\right) \sim \frac{MG\left(\frac{2}{r}-\frac{1}{a}\right)}{1+\frac{MG}{c^2}\left(\frac{2}{r}-\frac{1}{a}\right)}} \quad -(34)$$

6) Eq. (34) can be evaluated at the perihelion, where

$$r = \frac{\alpha}{1+\epsilon} \quad -(35)$$

This gives an expression for L in terms of the observables α, ϵ and x.

At the perihelion: $\quad -(36)$

$$\left(\frac{L}{m\alpha}\right)^2\left((1+x^2)(1+\epsilon)+x^2(\epsilon^2-1)\right)$$

$$= \frac{mG\left(\frac{2(1+\epsilon)}{\alpha}-\frac{(1-\epsilon^2)}{\alpha}\right)}{1+\frac{mG}{c^2}\left(\frac{2(1+\epsilon)}{\alpha}-\frac{(1-\epsilon^2)}{\alpha}\right)}$$

$$= \frac{\frac{mG}{\alpha}(1+\epsilon)^2}{1+\frac{mG}{c^2\alpha}(1+\epsilon)^2}$$

In the limit:

$$x \to 1 \quad -(37)$$

eq. (36) reduces to:

7) $$\left(\frac{L}{m\alpha}\right)^2 (1+\epsilon)^2 = \frac{MG}{\alpha}(1+\epsilon)^2 \quad -(38)$$

i.e.

$$L^2 = m^2 MG\alpha \quad -(39)$$

Q.E.D. This is a self consistent result.

In general:

$$\boxed{L_1^2 = m^2 MG\alpha y} \quad -(40)$$

where:

$$y = \frac{MG(1+\epsilon)^2}{\left(1+\frac{MG}{c^2\alpha}(1+\epsilon)\right)\left(1+x^2(1+\epsilon+\epsilon^2)\right)} \quad -(41)$$

at the perihelion.

It is seen that L_1^2 is a constant of motion and that this is a self consistent theory.

The observed change in angle at the perihelion is

$$\Delta\theta = 2\pi(x-1) \quad -(42)$$

and is explained by a change from L^2 to L_1^2