325(4): Einsteinian orbit by Numerical Integration

The hamiltonian is:

$$H = E = \frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 \right) + U \quad -(1)$$

where

$$U = -\frac{k}{r}\left(1 + \frac{L^2}{m^2 c^2 r^2}\right) \quad -(2)$$

Here

$$L = m r^2 \dot{\theta} \quad -(3)$$

and

$$k = m M G \quad -(4)$$

Make the change of variable:

$$\dot{r} = -\frac{L}{m}\frac{du}{d\theta} \quad -(5)$$

where

$$u = \frac{1}{r}. \quad -(6)$$

Therefore: $-(7)$

$$H = E = \frac{1}{2}\frac{L^2}{m}\left(\frac{du}{d\theta}\right)^2 + \frac{L^2 u^2}{2m} - ku\left(1 + \frac{L^2}{m^2 c^2} u^2\right)$$

i.e.

$$\frac{2m}{L^2} H = \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2km}{L^2} u \left(1 + \frac{L^2}{m^2 c^2} u^2\right)$$

$$= \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2u}{\alpha}\left(1 + \frac{L^2}{m^2 c^2} u^2\right) \quad -(8)$$

i.e.

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2m}{L^2} H - u^2 - \frac{2u}{\alpha}\left(1 + \frac{L^2}{m^2 c^2} u^2\right) \quad -(9)$$

1) So

$$\boxed{\theta = \int \left(\frac{2mH}{L^2} - u^2 + \frac{2u}{\alpha}\left(1 + \frac{L^2}{m^2c^2}u^2\right)\right)^{-1/2} du} \quad -(10)$$

The Newtonian integral is:

$$\theta = \int \left(\frac{2mH}{L^2} - u^2 + \frac{2u}{\alpha}\right)^{-1/2} du \quad -(11)$$

and gives the result:

$$u = \frac{1}{\alpha}\left(1 + \epsilon \cos\theta\right) \quad -(12)$$

where

$$\alpha = \frac{L^2}{m^2 MG} \quad \text{and} \quad \epsilon = \left(1 \pm \frac{2HL^2}{mk^2}\right)^{1/2}. \quad -(13)$$

Eq. (10) can be written as:

$$\theta = \int \left(N(u) + r_0 u^3 \right)^{-1/2} du \quad -(14)$$

where

$$N(u) = \frac{2mH}{L^2} - u^2 + \frac{2u}{\alpha} \quad -(15)$$

The integral (10) can be written as:

3)

$$\theta = \int \left(\frac{2mH}{L^2} - u^2 + \frac{2u}{\alpha} + r_0 u^3 \right)^{-1/2} du \qquad -(16)$$

$$= \int \left(\frac{2mH}{L^2} \left(1 - \frac{L^2}{2mH} \left(\frac{2u}{\alpha} - u^2 + r_0 u^3 \right) \right) \right)^{-1/2} du$$

If $\frac{2u}{\alpha} - u^2 + r_0 u^3 \ll \frac{L^2}{2mH}$ $\qquad -(17)$

then:

$$\theta \sim \left(\frac{L^2}{2mH} \right)^{1/2} \int \left(1 + \frac{L^2}{4mH} \left(\frac{2u}{\alpha} - u^2 + r_0 u^3 \right) \right) du \qquad -(18)$$

The Einsteinian effect in this limit is:

$$\Delta\theta \sim \left(\frac{L^2}{2mH} \right)^{1/2} \left(\frac{L^2}{4mH} \right) r_0 \int u^3 du \qquad -(19)$$

$$\Delta\theta = \frac{1}{2} \left(\frac{L^2}{2mH} \right)^{3/2} \cdot \frac{2mG}{c^2} \cdot \frac{u^4}{4} \qquad -(20)$$

$$= \frac{1}{8} \left(\frac{L^2}{2mH} \right)^{3/2} \left(\frac{2mG}{c^2} \right) \frac{1}{r^4}$$

4) Now use:

$$\frac{L^2}{2mE} = \frac{\alpha^2}{1-\epsilon^2} \quad -(21)$$

for an ellipse, and at the perihelion:

$$r = \frac{\alpha}{1+\epsilon} \quad -(22)$$

and

$$r = a(1-\epsilon) \quad -(23)$$

so

$$\Delta\theta = \frac{1}{8} \cdot \frac{2mG}{c^2 a} \frac{(1-\epsilon^2)^{3/2}}{(1-\epsilon)^4} \quad -(24)$$

This is not however the experimental result. This brings into question the validity of the Einstein theory.