325(6): True Orbit of the Sommerfeld Hamiltonian

Consider the Sommerfeld Hamiltonian:

$$H = (\gamma - 1) mc^2 + U \quad -(1)$$

$$= \left(\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right) mc^2 + U$$

where

$$U = -\frac{mMG}{r} \quad -(2)$$

Therefore:

$$\frac{H}{mc^2} - \frac{U}{mc^2} + 1 = \left(1 - \frac{v^2}{c^2} \right)^{-1/2}$$

$$= 1 + \frac{H - U}{mc^2} \quad -(3)$$

$$= (H + mc^2 - U) / mc^2$$

So

$$1 - \frac{v^2}{c^2} = \left(\frac{mc^2}{H - U + mc^2} \right)^2 \quad -(4)$$

and

$$\frac{v^2}{c^2} = 1 + \left(\frac{mc^2}{H - U + mc^2} \right)^2$$

$$= \frac{\frac{v_0^2}{c^2}}{1 + \frac{v_0^2}{c^2}} \quad -(5)$$

a) where:

$$V_0 = \frac{L^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad -(6)$$

Therefore:

$$\frac{v_0^2}{c^2} = \left(1 + \frac{v_0^2}{c^2}\right)\left(1 + \left(\frac{mc^2}{H + \frac{mMG}{r} + mc^2}\right)\right) \quad -(7)$$

Using

$$u = \frac{1}{r} \quad -(8)$$

then

$$V_0 = \frac{L^2}{m^2}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \quad -(9)$$

and

$$\frac{v_0^2}{c^2} = \left(1 + \frac{v_0^2}{c^2}\right)\left(1 + \frac{mc^2}{H + mMGu + mc^2}\right) \quad -(10)$$

Computer Algebra

Solve eq. (10) for $du/d\theta$ and find

$$\frac{d\theta}{du} = f(u) \quad -(11)$$

then

$$\theta = \int f(u)\, du \quad -(12)$$

This will give θ as a function of u. Compare with the result for an ellipse, which is:

$$\theta_0 = \cos^{-1}\left[\frac{\left(\frac{L^2}{m^2 M G}\frac{1}{r} - 1\right)}{1 + \frac{2HL^2}{m k^2}}\right] \quad -(13)$$

where

$$k = mMG \quad -(14)$$

The precession is:

$$\Delta\theta = \theta - \theta_0 \quad -(15)$$

Comparison of Velocity Curves

In the Newtonian limit:

$$v_0^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(16)$$

so

$$\frac{d(v_0^2)}{dr} = -\frac{2MG}{r^3} \quad -(17)$$

The relativistic velocity is

$$v^2 = \frac{v_0^2}{1 + \frac{v_0^2}{c^2}} \quad -(18)$$

so

$$\frac{d(v^2)}{d(v_0^2)} = \frac{2v_0\left(1 + \frac{v_0^2}{c^2}\right) - \frac{2v_0^3}{c^2}}{\left(1 + \frac{v_0^2}{c^2}\right)^2} \quad -(19)$$

4) Therefore:

$$\frac{d(v^2)}{dr} = \frac{d(v_0^2)}{dr} \cdot \frac{d(v^2)}{d(v_0^2)} \quad -(19)$$

1) Compare the plot of $d(v^2)/dr$ with the plot of $d(v_0^2)/dr$

2) Repeat for the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(20)$$

Compare (1) and (2) to find how closely eq. (20) compares with the true curve from the Sommerfeld hamiltonian. In this way the true Sommerfeld hamiltonian can be found.

precession of the Sommerfeld hamiltonian can be found.