325(5): Calculation of the Relativistic Hamiltonian for Various Orbits

The relativistic hamiltonian of special relativity is

$$H = \gamma m c^2 + U \quad - (1)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (2)$$

Here U is the potential energy and v is the velocity of one frame with respect to another. In orbital theory m is a mass orbiting a mass M, and U is the potential energy defined by the force between m and M:

$$F = -\frac{dU}{dr} \quad - (3)$$

The velocity v is defined by the Minkowski metric:

$$c^2 d\tau^2 = (c^2 - v^2) dt^2 \quad - (4)$$

where $d\tau$ is the infinitesimal of proper time and where dt is the infinitesimal of time in the observer frame. In eq. (4):

$$|d\underline{r}|^2 = v^2 dt^2, \quad - (5)$$

so

$$v^2 = \left(\frac{|d\underline{r}|}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 r^2 \quad - (6)$$

2)

From a Lagrangian analysis:

$$\frac{d\theta}{dt} = \frac{L}{\gamma m r^2} \quad - (7)$$

and

$$\frac{dr}{dt} = -r^2 \frac{du}{d\theta}\frac{d\theta}{dt} \quad - (8)$$

where

$$u = 1/r \quad - (9)$$

So

$$\frac{dr}{dt} = -r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right) \cdot \frac{L}{\gamma m r^2}$$

$$= -\frac{L}{\gamma m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad - (10).$$

It follows that:

$$v^2 = \left(\frac{L}{\gamma m}\right)^2 \left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad - (11)$$

$$= \left(\frac{L}{m}\right)^2 \left(1 - \frac{v^2}{c^2}\right)\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)$$

i.e

$$v^2 = \frac{\frac{L^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)}{1 + \frac{L^2}{m^2c^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)} \quad - (12)$$

3) The hamiltonian (1) may be rewritten as:

$$H - mc^2 = (\gamma - 1) mc^2 + U \quad - (13)$$

In the non relativistic limit:

$$v << c, \quad - (14)$$

$$H_1 = H - mc^2 \longrightarrow \frac{1}{2} mv^2 + U = \frac{p^2}{2m} + U \quad - (15)$$

therefore H_1 corresponds to the familiar non-relativistic hamiltonian, a constant of motion. For any valid orbit:

$$H_1 = (\gamma - 1) mc^2 + U \quad - (16)$$

must be a constant, where γ is defined by eqs. (2) and (12).

<u>First Approximation</u>: <u>The Newtonian Orbit</u>

The Newtonian orbit is non-relativistic and is valid only under the condition (14). It will not therefore result in a constant H_1. It can be regarded as the first step in an iterative procedure that will eventually result in the correct orbit for eq. (16), one in which H_1 is a constant.

4) So the first step in the calculation is to find H_1 for the Newtonian orbit defined by:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad - (17)$$

and

$$U = -mMG/r \quad - (18)$$

Therefore:

$$H_1(\text{Newton}) = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} - \frac{mMG}{r} \quad - (19)$$

where

$$v^2 = \frac{L^2}{m^2}\left[\frac{1+\epsilon^2+2\epsilon\cos\theta}{\alpha^2+\frac{L^2}{m^2c^2}(1+\epsilon^2+2\epsilon\cos\theta)}\right] - (20)$$

and

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right) - (21)$$

Therefore $H_1(\text{Newton})$ can be worked out with computer algebra, giving the result:

$$H_1 = H(\text{true}) + f(r) \quad -(22)$$

where $f(r)$ is a function of r, and also of the observables ϵ and α of the orbit.

5) The true hamiltonian is therefore:

$$H(\text{true}) = (\gamma - 1) m c^2 + U - f(r) \quad -(23)$$

and the relativistic potential for a Newtonian orbit is

$$V = U - f(r)$$
$$= -\frac{mMG}{r} - f(r) \quad -(24)$$

and the true force is:

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} + \frac{\partial f(r)}{\partial r} \quad -(25)$$

The relativistic orbit from a central potential (18) can be found by assuming that the relativistic kinetic energy in eq. (23) is changed to:

$$T(\text{true}) = (\gamma - 1) m c^2 - f(r) \quad -(26)$$
$$= T(\text{Newton}) - f(r)$$

This calculation can now be repeated for the precessing orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(27)$$

which will be the subject of the next note.