324(2): Details of the Lagrangian Calculations

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(1)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(2)$$

with

$$\mathcal{L} = -mc^2 f^{1/2} + \frac{MG}{r} \quad -(3)$$

where

$$f = 1 - \frac{1}{c^2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right) \quad -(4)$$

From differential algebra: $-(5)$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial r}, \quad \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \dot{r}}, \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \dot{\theta}}$$

From eq. (4):

$$\frac{\partial f}{\partial r} = -\frac{2r\dot{\theta}^2}{c^2} \quad -(6)$$

$$\frac{\partial f}{\partial \dot{r}} = -\frac{2\dot{r}}{c^2} \quad -(7)$$

$$\frac{\partial f}{\partial \dot{\theta}} = -\frac{2r^2\dot{\theta}}{c^2} \quad -(8)$$

So from eqs. (5) and (6), wits:

2)
$$\frac{\partial \mathcal{L}}{\partial f} = -\frac{mc^2}{2} f^{-1/2} \quad -(9)$$

it follows that
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{mc^2}{2} f^{-1/2} \cdot \frac{2r\dot{\theta}^2}{c^2} - \frac{\partial U}{\partial r}$$
$$= m f^{-1/2} r \dot{\theta}^2 - \partial U/\partial r \quad -(10)$$
$$= \gamma m r \dot{\theta}^2 - \frac{\partial U}{\partial r}.$$

Similarly:
$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{mc^2}{2} f^{-1/2} \cdot \frac{2\dot{r}}{c^2} = \gamma m \dot{r} \quad -(11)$$

So
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \gamma m \ddot{r} \quad -(12)$$

Fm eqs. (2), (10) and (12):
$$\gamma m r \dot{\theta}^2 - \frac{\partial U}{\partial r} = \gamma m \ddot{r} \quad -(13)$$

So
$$\boxed{\gamma\left(m\ddot{r} - m r \dot{\theta}^2\right) = -\frac{\partial U}{\partial r} = F(r)} \quad -(14)$$

Q.E.D.

Similarly:
$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad -(15)$$

3) and

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial f} \frac{\partial f}{\partial \dot{\theta}} \quad -(16)$$

$$= \frac{mc^2}{2} f^{-1/2} \cdot \frac{2r^2\dot{\theta}}{c^2}$$

$$= \gamma m r^2 \dot{\theta}$$

$$:= L$$

where

$$L = \gamma m r^2 \dot{\theta} \quad -(17)$$

is the relativistic angular momentum. This is a conserved quantity:

$$\frac{dL}{dt} = 0 \quad -(18)$$

from eq. (1), Q.E.D.

Results

$$\gamma m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \quad -(19)$$

$$L = \gamma m r^2 \dot{\theta} \quad -(20)$$

$$\frac{dL}{dt} = 0 \quad -(21)$$