324(6): Self Consistency of the Relativistic Binet and Lorentz Force Equations.

For a planar orbit the relativistic hamiltonian can be written as:

$$H - mc^2 = \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) mc^2 + U \quad -(1)$$

where:

$$v^2 = \frac{\frac{L^2}{m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)}{1 + \frac{L^2}{m^2c^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)} \quad -(2)$$

The relativistic Binet equation is therefore:

$$F = -\frac{\partial U}{\partial r} = \frac{\partial}{\partial r}\left((\gamma - 1) mc^2\right) \quad -(3)$$

because $H - mc^2$ is a constant of motion. The integral form of the relativistic Binet equation is eq. (1).

In these equations:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

2) where γ^2 is given by eq. (2). So having found the force and potential from eq. (3), the shifted Hamiltonian $H - mc^2$ can be found from eq. (1).

These equations apply to any planar orbit, so the gravitational force, potential and shifted Hamiltonian can be found for any planar, relativistic orbit.

This analysis is equivalent to to the ECE 2 Lorentz force equation for any boost:

$$\underline{F} = m\left(\gamma\left(\underline{g} + \underline{v} \times \underline{\Omega}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c}\cdot\underline{g}\right)\right) \quad -(5)$$

because the method of special relativity have been used both for eq. (1) and eq. (5). Therefore for a planar orbit the force (3) and the force from eq. (5) must be the same.

For a planar orbit the force from eq. (3) is:

$$\underline{F} = -\left(\frac{\partial U}{\partial r}\right)\underline{e}_r = \left(\frac{\partial}{\partial r}\left((\gamma-1)mc^2\right)\right)\underline{e}_r \quad -(6)$$

Therefore we obtain the general solution of eq. (5):

$$\underline{F} = m\left(\gamma(\underline{g} + \underline{v} \times \underline{\Omega}) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{g}\right)\right)$$

$$= \left(\frac{\partial}{\partial r}\left((\gamma - 1)mc^2\right)\right)\underline{e}_r \quad -(7)$$

for any orbit in a plane. Here γ is given by eq. (4) and v^2 by eq. (2). This is a fully self consistent relativistic analysis.

Examples

1) Precessing orbit:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(8).$$

2) Hyperbolic spiral orbit in a whirlpool galaxy:

$$r = \frac{r_0}{\theta} \quad -(9)$$

The relativistic force for each orbit is given by eq. (7). Computer algebra can be used to work out F, U and $H - mc^2$ for each orbit, and the results graphed.

The special case of the Newtonian orbit has been worked out in notes 324(4) and 324(5), giving:

$$U = -mMG/r, \quad F = -mMG/r^2, \quad |H| = |E| = \frac{mMG}{2a}$$

$$-(10)$$