324(5): The Classical, non-relativistic, theory of a Precessing Orbit

The phenomena of a precessing orbit can be understood without any use of relativity as follows. Obviously this means that the Einstein theory is not needed, and the simpler classical treatment is preferred by Ockham's Razor. The only argument for a relativistic theory is that the field equations of ECE2 are Lorentz covariant, and that special relativity is very precise in other contexts such as time dilatation.

Consider the precessing conical section orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(1)$$

where x is very close to unity. The central force law for this orbit is:

$$\underline{F} = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \quad -(2)$$

which is the Binet equation of orbits. Therefore:

$$F = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(3)$$

where

$$L = mr^2\dot{\theta} \quad -(4)$$

is the conserved angular momentum defined by:

2)

$$\frac{dL}{dt} = 0 \quad -(5)$$

so L is a constant of motion. The other constant of motion is the Hamiltonian H. In eq. (1) α is the half right latitude and ϵ is the ellipticity. The orbit is that of a mass m orbiting a mass M.

From eqs. (1) and (3):

$$F = -\frac{\partial U}{\partial r} = -x^2\frac{L^2}{mr^2\alpha} + (x^2-1)\frac{L^2}{mr^3} \quad -(6)$$

This force is a combination of the Hooke Newton inverse square law and a term in inverse cube r. The Einstein theory gives an incorrect combination of terms in inverse r squared and inverse fourth power of r. This was first pointed out a few years ago. Therefore the gravitational potential is:

$$U = -\frac{x^2 L}{m\alpha r} + \frac{1}{2}(x^2-1)\frac{L^2}{mr^2} \quad -(7)$$

When

$$x = 1 \quad -(8)$$

eqs. (6) and (7) reduce to the Newtonian results:

$$F = -\frac{mMG}{r^2}, \quad U = -\frac{mMG}{r} \quad -(8)$$

) using the Newtonian:

$$L^2 = m^2 MG\alpha \quad -(9)$$

Using the newly discovered integral form of the Binet equation the hamiltonian can be calculated as follows:

$$H = U + \frac{L^2}{2m}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad -(10)$$

From eqs. (1), (7) and (10):

$$\boxed{H = \frac{x^2 L^2}{2m}\left(\frac{\epsilon^2 - 1}{\alpha^2}\right)} \quad -(11)$$

the Newtonian result is obtained when $x = 1$

from eq. (11)

$$H(\text{Newtonian}) = \frac{L^2}{2m}\left(\frac{\epsilon^2-1}{\alpha^2}\right) \quad -(12)$$

and

$$\frac{dH}{dt} = 0 \quad -(13)$$

so H is a constant of motion, Q.E.D.

Therefore precession produces a shift in the hamiltonian of the orbit. This is a new and clear result for the integral form of the Binet equation.

4) This is a complete and sufficient classical theory. The Lorentz force method on the other hand is part of a generally covariant unified field theory, ECE2 theory, whose field equations are mathematically Lorentz covariant. In the limit:

$$v << c \quad -(14)$$

the Lorentz force of the ECE2 theory is:

$$\underline{F} = m\left(\frac{d^2 r}{dt^2}\,\underline{e}_r + \underline{v} \times \underline{\Omega}\right) \quad -(15)$$

For a planar orbit:

$$\underline{F} = m\left(\frac{d^2 r}{dt^2} - \Omega^2 r\right)\underline{e}_r \quad -(16)$$

Here $\underline{\Omega}$ is the gravitomagnetic field and $\underline{v}$ is the velocity of one frame with respect to another in the general Lorentz boost. So from eq. (16)

$$F = m\left(\frac{d^2 r}{dt^2} - \Omega^2 r\right) \quad -(17)$$

Comparing eqns. (16) and (17):

$$\frac{d^2 r}{dt^2} = \Omega^2 r - \frac{x^2 L^2}{m r^2 \alpha} + \frac{(x^2 - 1) L^2}{m r^3} \quad -(18)$$

which is the Leibnitz equation of the precessing orbit.

5) In the Newtonian limit:

$$x \rightarrow 1, \quad \Omega \rightarrow \omega \qquad -(19)$$

and

$$\frac{d^2 r}{dt^2} \longrightarrow \omega^2 r - \frac{mMG}{r^2} \qquad -(20)$$

The centrifugal force is:

$$F_c = m\omega^2 r \qquad -(21)$$

and the force of attraction is:

$$F_A = -\frac{mMG}{r^2} \qquad -(22)$$

In order to reduce eq. (15) to eq. (16):

$$\begin{aligned} \underline{v} \times \underline{\Omega} &= -\underline{\Omega} \times \underline{v} \\ &= -\underline{\Omega} \times (\underline{\Omega} \times \underline{r}) \qquad -(23) \\ &= -\Omega^2 r \, \underline{e}_r \end{aligned}$$

Therefore one frame rotates with respect to the other with orbital linear velocity:

$$\underline{v} = \underline{\Omega} \times \underline{r}. \qquad -(24)$$

In the classical limit of ECE2:

$$\underline{\Omega} \rightarrow \underline{\omega} = \frac{d\theta}{dt} \underline{k}, \qquad -(25)$$

a self consistent result.

6) In order to develop this classical theory into special relativity, the lagrangian must be replaced by:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(26)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(27)$$

and

$$v^2 = \dot{r}^2 + \omega^2 r^2 \quad -(28)$$
$$= \dot{r}^2 + \dot{\theta}^2 r^2$$

The two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(29)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(30)$$

The Hamiltonian is:

$$H = \gamma mc^2 - \frac{mMG}{r} \quad -(31)$$

i.e.

$$H - mc^2 = (\gamma - 1)mc^2 - \frac{mMG}{r} \quad -(32)$$

From eqs. (26), (29) and (30):

$$\dot{r} = -\frac{L}{\gamma m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(33)$$

7) and $\dot{\theta} = \frac{L}{\gamma m r^2}$ — (24)

so the integral form of the relativistic Binet equation is :

$$H - mc^2 = \left(\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right) mc^2 - \frac{mMG}{r} \quad -(25)$$

where
$$v^2 = \dot{r}^2 + \dot{\theta}^2 r^2 \quad -(26)$$
$$= \frac{L^2}{\gamma^2 m^2} \left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} \right)$$
$$= \frac{L^2}{m^2} \left(1 - \frac{v^2}{c^2} \right) \left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} \right)$$

so
$$v^2 = \frac{\frac{L^2}{m^2} \left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} \right)}{1 + \frac{1}{c^2} \left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} \right)} \quad -(27)$$

The hamiltonian may be expressed in terms of x using eqs. (1) and (25) and compared with the non-relativistic result (11)