324(3): A New Equation of Orbits

In plane polar coordinates the fundamental equations are:

$$H = E = \frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 \right) + U \quad -(1)$$

and

$$F = m \left(\ddot{r} - r \dot{\theta}^2 \right) = - \frac{\partial U}{\partial r} \quad -(2)$$

When the total energy E is zero:

$$\frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 \right) = - U \quad -(3)$$

The angular momentum is a constant of motion, so:

$$L = m r^2 \dot{\theta} = \text{constant} \quad -(4)$$

Using

$$\dot{\theta} = \frac{L}{m r^2} \quad -(5)$$

it is found that:

$$\dot{r} = - \frac{L}{m} \frac{d}{d\theta} \left(\frac{1}{r} \right) \quad -(6)$$

(previous UFT papers and notes on the Binet equation of orbits).

Using eqs. (4) and (6), eq. (3) becomes:

$$\boxed{U = - \frac{L^2}{2m} \left(\left(\frac{d}{d\theta} \left(\frac{1}{r} \right) \right)^2 + \frac{1}{r^2} \right)} \quad -(7)$$

2) This is a new equation of orbits. The force for any orbit is:

$$F = -\frac{\partial U}{\partial r} \quad -(8)$$

Conic Section Orbits

In this case:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(9)$$

So:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\sin\theta \quad -(10)$$

and

$$\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 = \frac{\epsilon^2}{\alpha^2}\left(1-\cos^2\theta\right) \quad -(11).$$

From eq. (9):

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right) \quad -(12)$$

So

$$\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 = \frac{\epsilon^2}{\alpha^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{\alpha^2}{r^2}-\frac{2\alpha}{r}+1\right)\right)$$

$$= \frac{2}{\alpha r} - \frac{1}{r^2} \quad -(13)$$

From eqs. (7) and (13):

$$U = -\frac{L^2}{m\alpha r} \quad -(14)$$

For conical section orbits:

$$L^2 = m^2 MG\alpha \quad -(15)$$

So

$$U = -\frac{mG}{r} \quad -(16)$$

which is the gravitational potential Q.E.D.

Eq. (7) is valid for any planar orbit, and it is concluded that any planar orbit in equilibrium is described by:

$$E = 0. \quad -(17)$$

The relativistic generalization of eq. (7) is obtained from:

$$H = E = \gamma mc^2 + U, \quad -(18)$$

and the Lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(19)$$

Eq. (18) is conveniently written as the Sommerfeld hamiltonian:

4)
$$E - mc^2 = (\gamma - 1)mc^2 + U \quad -(20)$$
where the relativistic kinetic energy is:
$$T = (\gamma - 1)mc^2 \quad -(21)$$
The Lorentz factor is:
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(22)$$
and when
$$v \ll c \quad -(23)$$
$$\gamma \rightarrow 1 + \frac{1}{2}\frac{v^2}{c^2} \quad -(24)$$
so
$$T \rightarrow \frac{1}{2}mv^2 \quad -(25)$$
For orbits in equilibrium:
$$v^2 = \dot{r}^2 + \dot{\theta}^2 r^2 \quad -(26)$$
The relativistic generalization of eq. (7) is obtained when:
$$E - mc^2 = 0 \quad -(27)$$
i.e when
$$E = mc^2, \quad -(28)$$
the rest energy.

So the relativistic equation of orbits is:

5)

$$\boxed{U = -(\gamma - 1)mc^2} \quad -(29)$$

where

$$L = \gamma m r^2 \dot{\theta} \quad -(30)$$

is a constant of motion.

Eq. (29) is generally valid for any relativistic orbit. Here:

$$\gamma = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right)\right)^{-1/2} \quad -(31)$$

where

$$\dot{r} = -\frac{L}{\gamma m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(32)$$

and

$$\dot{\theta} = \frac{L}{\gamma m r^2} \quad -(33)$$

Computer algebra can be used to find the relativistic U for the precessing orbit:

$$\alpha = \frac{r}{1 + \epsilon \cos(x\theta)} \quad -(34)$$

The relativistic generalization of eq. (2) is as follows:

6)

$$\frac{d}{dt}\left(\gamma m \dot{r}\right) - \gamma m r \dot{\theta}^2 = -\frac{\partial U}{\partial r} = F(r) \quad -(35)$$

In which:

$$m \frac{d}{dt}\left(\gamma \dot{r}\right) = m\left(\dot{r}\frac{d\gamma}{dt} + \gamma \ddot{r}\right) \quad -(36)$$

and

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad -(37)$$

Fm eq. (22)

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(38)$$

So

$$\frac{d\gamma}{dt} = \gamma^3 \frac{v}{c^2}\frac{dv}{dt} \quad -(39)$$

Therefore:

$$\frac{d}{dt}\left(\gamma m \dot{r}\right) = m\left(\dot{r}\gamma^3 \frac{v}{c^2}\frac{dv}{dt} + \gamma \ddot{r}\right) \quad -(40)$$

where

$$v = \left(\dot{r}^2 + \dot{\theta}^2 r^2\right)^{1/2} \quad -(41)$$

and where $\dot{r}$ and $\dot{\theta}$ are given by eqs. (32) and (33).

This procedure leads to the relativistic Binet equation, but the new eq. (29) is much simpler.