324(1): The Relativistic Binet Equation.

First consider the relativistic lagrangian defined by:

$$p = \frac{\partial \mathcal{L}}{\partial v} \quad -(1)$$

where p is the relativistic momentum:

$$\underline{p} = \gamma m \underline{v}. \quad -(2)$$

As in Marion and Thornton page 539, the relativistic lagrangian is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(3)$$

$$= -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} - U.$$

The relativistic hamiltonian is defined by:

$$H = vp - \mathcal{L}$$

$$= \gamma m v^2 - \mathcal{L}$$

$$= \frac{p^2}{\gamma m} + \frac{mc^2}{\gamma} + U \quad -(4)$$

$$= E + U$$

where

$$E = \gamma m c^2. \quad -(5)$$

Now consider the Euler Lagrange equations with the relativistic lagrangian (3):

2)
$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad -(6)$$

and
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \quad -(7)$$

In plane polar coordinates the velocity is defined by:
$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(8)$$

so
$$v^2 = \dot{r}^2 + \omega^2 r^2 \quad -(9)$$

The Lorentz factor is therefore:
$$\gamma = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right)\right)^{-1/2} \quad -(10)$$

Therefore:
$$\mathcal{L} = -mc^2\left(1 - \frac{1}{c^2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right)\right)^{1/2} - U \quad -(11)$$

For a central inverse square attraction between m orbiting M:
$$U = -\frac{MG}{r} \quad -(12)$$

so
$$\mathcal{L} = -mc^2 f^{1/2} + \frac{mG}{r} \quad -(13)$$

where:

3)

$$f := 1 - \frac{1}{c^2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right) \quad -(14)$$

Therefore:

$$\boxed{L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \gamma m r^2 \dot{\theta}} \quad -(15)$$

and

$$\frac{dL}{dt} = 0 \quad -(16)$$

Therefore L is the conserved relativistic angular momentum.

From eq. (7):

$$\boxed{\gamma m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r) = -\frac{\partial U}{\partial r}} \quad -(17)$$

Eq. (17) is the one to be compared with the Lorentz force equation of ECE2 general relativity:

$$\underline{F} = \gamma m\left(\underline{a}_N + \underline{v} \times \underline{\Omega}\right) - \frac{\gamma^2}{1+\gamma}\frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{a}_N\right) \quad -(18)$$

where the Newtonian acceleration is:

$$\underline{a}_N = \frac{d^2 r}{dt^2}\underline{e}_r = \ddot{r}\,\underline{e}_r \quad -(19)$$

4) Eq. (18) is more general than eq. (17) because the latter has assumed orbital motion in a plane, whereas eq. (18) is true for general dynamics. However, both equations are equations of general relativity.

Eqs. (15) and (17) can be rewritten as the generally covariant Binet equation for planar orbits. Consider:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \quad -(20)$$

where

$$\frac{d\theta}{dt} = \frac{L}{\gamma m r^2} \quad -(21)$$

from eq. (15). So:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\gamma m}{L}\frac{dr}{dt} \quad -(22)$$

Similarly:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\gamma m}{L}\frac{d}{d\theta}\frac{dr}{dt}$$

$$= -\frac{\gamma m}{L}\frac{dt}{d\theta}\frac{d^2 r}{dt^2}$$

$$= -\frac{\gamma^2 m^2 r^2}{L^2}\frac{d^2 r}{dt^2} \quad -(23)$$

From eq. (21):

5)

$$r\dot{\theta}^2 = \frac{L^2}{\gamma^2 m^2 r^3} \quad -(24)$$

Using eqns. (23) and (24) in eq. (17):

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma m r^2}{L^2} F(r) \quad -(25)$$

and:

$$\boxed{F(r) = -\frac{L^2}{\gamma m r^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)} \quad -(26)$$

This is the relativistic Binet equation, Q.E.D.

It is equivalent to:

$$F(r) = \gamma m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U}{\partial r} \quad -(27)$$

However eq. (26) can be used to calculate the relativistic force for any planar orbit, i.e. an observed orbit.

Eq. (26) can be used to calculate $F(r)$ using:

6)

$$F(r) = \frac{L^2}{mr^2}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right) \quad -(28)$$

in which:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(29)$$

$$= \left(\frac{d\theta}{dt}\right)^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)$$

$$= \frac{L^2}{\gamma^2 m^2 r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)$$

$$= \frac{L^2}{mr^2}\left(1-\frac{v^2}{c^2}\right)\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(30)$$

So

$$\boxed{v^2 = \frac{L^2}{m^2 r^4}\left(1+\frac{L^2}{m^2c^2r^4}\left(r^2+\left(\frac{dr}{d\theta}\right)^2\right)\right)^{-1}}$$

The observed non-Newtonian orbit is the precessing conical section, for example the precessing ellipse, where:

7)
$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(31)$$

At the perihelion, r_{min}:
$$\alpha = (1+\epsilon)\, r_{min} \quad -(32)$$

The relativistic force can be calculated by computer algebra using eqns (28), (30) and (31), and it can be plotted against the inverse square law:
$$F = -\frac{mMG}{r^2} \quad -(33)$$

for different x.

From a comparison of eqs. (17) and (18) for planar orbits:
$$\gamma m(\ddot{r} - r\dot{\theta}^2)\underline{e}_r = \gamma m(\underline{a}_N + \underline{v}\times\underline{\Omega}) \quad -(34)$$

so
$$\boxed{(\ddot{r} - r\dot{\theta}^2)\underline{e}_r = \underline{a}_N + \underline{v}\times\underline{\Omega}} \quad -(35)$$

in this relativistic theory.