324(4): The Integral Form of Binet's Equation.

Consider the Hamiltonian in plane polar coordinates:

$$H = E = T + U \qquad -(1)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 \right) + U$$

The Hamiltonian H is a constant of motion. The angular momentum L is also a constant of motion:

$$L = m r^2 \dot{\theta} = \text{constant.} \qquad -(2)$$

From eq. (2) it is found that:

$$\dot{r} = - \frac{L}{m} \frac{d}{d\theta} \left(\frac{1}{r} \right) \qquad -(3)$$

(previous papers and notes on the Binet equation of orbits). From eq. (2):

$$\dot{\theta} = \frac{L}{m r^2} \qquad -(4)$$

From eqs. (1), (3) and (4): $\qquad -(5)$

$$\boxed{U = H - \frac{L^2}{2m} \left(\left(\frac{d}{d\theta} \left(\frac{1}{r} \right) \right)^2 + \frac{1}{r^2} \right)}$$

This is the integral form of Binet's equation. It can be used to calculate U and H for any orbit and is a new equation of orbits.

d) For example, for the conic-section orbits:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(6)$$

computer algebra shows that:

$$\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 = \frac{2}{\alpha r} - \frac{1}{r^2} + \frac{1}{\alpha^2}\left(\epsilon^2 - 1\right) \quad -(7)$$

So

$$U = H - \frac{mMG}{r} - \frac{L^2}{2m\alpha^2}\left(\epsilon^2 - 1\right) \quad -(8)$$

The experimentally observed potential for an inverse square force law is:

$$U = -\frac{mMG}{r} \quad -(9)$$

so

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(10)$$

So from eq. (8):

$$H = \frac{L^2}{2m\alpha^2}\left(\epsilon^2 - 1\right) \quad -(11)$$

$$= \text{constant} \quad -(12)$$

Therefore

$$\boxed{\frac{\epsilon^2 - 1}{\alpha^2} = \text{constant of motion}}$$

). For an ellipse the semi major axis is:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(13)$$

and this is a constant of the orbit. The half right latitude α is also a constant of the orbit. For a conical section orbit:

$$L^2 = m^2 M G \alpha \quad -(14)$$

and

$$\frac{\epsilon^2 - 1}{\alpha^2} = \frac{1}{a\alpha} \quad -(15)$$

$$|H| = \frac{mMG}{2a} = |E| \quad -(16)$$

This is eq. (7.42), page 258, of J.B. Marion and S.T. Thornton, "Classical Dynamics" (HB College Publishing, 3rd edition, 1988).

Therefore eq. (5) is correct and self consistent, Q.E.D.

It can now be applied to other types of planar orbit. For examples:

4)

1) The precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \; ; \quad -(17)$$

2) The logarithmic spiral:

$$r = r_0 \exp(k\theta) \quad -(18)$$

and 3) The hyperbolic spiral of whirlpool galaxies:

$$r = \frac{r_0}{\theta} \quad -(19)$$

Relativistic Generalization

The hamiltonian is:

$$H = \gamma m c^2 + U \quad -(20)$$

or

$$H - mc^2 = (\gamma - 1) mc^2 + U \quad -(21)$$

Eq. (21) is the Sommerfeld hamiltonian used in atomic theory.

As in previous notes:

$$\dot{r} = -\frac{L}{\gamma m} \frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(22)$$

and

$$\dot{\theta} = \frac{L}{\gamma m r^2} \quad -(23)$$

5) The Lorentz factor is

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(24)$$

where

$$v^2 = \dot{r}^2 + \dot{\theta}^2 r^2 \quad -(25)$$

From eqs. (22), (23) and (25):

$$v^2 = \frac{L^2}{\gamma^2 m^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad -(26)$$

So the Sommerfeld hamiltonian (21) is:

$$H - mc^2 = mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) + U \quad -(27)$$

with v^2 defined by eq. (26). The latter is:

$$v^2 = \frac{L^2}{m^2}\left(1 - \frac{v^2}{c^2}\right)\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right) \quad -(28)$$

so

$$v^2 = \frac{L^2}{m^2}\left[1 + \frac{1}{c^2}\left(\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right)\right]^{-1} \quad -(29)$$

6) From eq. (27):

$$\bar{U} = H - mc^2 - mc^2\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right) \quad -(30)$$

where:

$$v^2 = \frac{L^2}{m^2}\left[1 + \frac{1}{c^2}\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2}\right]^{-1} \quad -(31)$$

The force is:

$$F = -\frac{\partial \bar{U}}{\partial r} = mc^2 \frac{\partial}{\partial r}\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right) \quad -(32)$$

and the relativistic orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(33)$$

The force F can be worked out from eqs. (31) to (33) by computer algebra and plotted. The same force can be calculated from the Lorentz transform of ECE2.