Note 320(7): Complete Solution of the Planar Orbital Problem

The orbital velocity in general is:

$$\underline{v} = \dot{r}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(1)$$

and the orbital acceleration is:

$$\underline{a} = \ddot{r}\,\underline{e}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(2)$$

Therefore the orbital force is:

$$\underline{F} = m\underline{g} + \underline{v}_{rot} \times \underline{\omega} \quad -(3)$$

where

$$\underline{v}_{rot} = \underline{\omega} \times \underline{r} \quad -(4)$$

The ECE2 Lorentz force is:

$$\underline{F} = m\underline{g} + \underline{v}_L \times \underline{\Omega} \quad -(5)$$

where $\underline{\Omega}$ is the gravitomagnetic field. It follows directly from eqs. (3) and (5) that

$$\underline{\Omega} = \underline{\omega} = \omega\,\underline{k} = \frac{d\theta}{dt}\,\underline{k} \quad -(6)$$

and

$$\underline{v}_L = \underline{\omega} \times \underline{r} \quad -(7)$$

The gravitomagnetic field equations of ECE2 in general are pseudo Lorentz covariant, giving eq. (3) in the non-relativistic limit, and are:

2) More generally the electromagnetic Lorentz transforms are:

$$\underline{E}' = \underline{E} + \underline{v} \times \underline{B} \quad -(8)$$

$$\underline{B}' = \underline{B} - \frac{1}{c^2} \underline{v} \times \underline{E} \quad -(9)$$

and the gravitational Lorentz transforms are:

$$\underline{g}' = \underline{g} + \underline{v} \times \underline{\Omega} \quad -(10)$$

$$\underline{\Omega}' = \underline{\Omega} - \frac{1}{c^2} \underline{v} \times \underline{g} \quad -(11)$$

The electromagnetic Biot Savart law is obtained from

$$\underline{B} = \underline{0} \quad -(12)$$

in eq. (9) and is:

$$\underline{B}' = -\frac{1}{c^2} \underline{v} \times \underline{E} \quad -(13)$$

The Biot Savart law (13) is usually written as:

$$\underline{B} = -\frac{1}{c^2} \underline{v} \times \underline{E} \quad -(14)$$

and is equivalent to the Ampère law:

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(15)$$

It follows that:

$$\underline{\nabla} \times \underline{B} = -\frac{1}{c^2}\,\underline{\nabla} \times (\underline{v} \times \underline{E}) = \mu_0 \underline{J} \quad -(16)$$

so the current density of electrodynamics is:

$$\underline{J} = -\frac{1}{\mu_0 c^2}\,\underline{\nabla} \times (\underline{v} \times \underline{E}) \quad -(17)$$

where

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \quad -(18)$$

so

$$\underline{J} = -\epsilon_0\,\underline{\nabla} \times (\underline{v} \times \underline{E}) \quad -(19)$$

Here:

$$\underline{\nabla} \times (\underline{v} \times \underline{E}) = \underline{v}(\underline{\nabla} \cdot \underline{E}) - (\underline{\nabla} \cdot \underline{v})\underline{E} + (\underline{E} \cdot \underline{\nabla})\underline{v} - (\underline{v} \cdot \underline{\nabla})\underline{E} \quad -(20)$$

The electromagnetic charge current four density is:

$$J^\mu = (c\rho, \underline{J}) \quad -(21)$$

The gravitational Biot Savart law is:

$$\boxed{\underline{\Omega} = -\frac{1}{c^2}\,\underline{v} \times \underline{g}} \quad -(22)$$

and is equivalent to:

$$\underline{\nabla} \times \underline{\Omega} = \frac{4\pi G}{c^2}\,\underline{J}_m \quad -(23)$$

4) where

$$J_m^\mu = (c\rho_m, \underline{J}_m) \quad -(24)$$

is the gravitational mass / current density.

Therefore:

$$\underline{\nabla} \times \underline{\Omega} = -\frac{1}{c^2} \underline{\nabla} \times (\underline{v} \times \underline{g}) = \frac{4\pi G}{c^2} \underline{J}_m \quad -(25)$$

and

$$\boxed{\underline{J}_m = -\frac{1}{4\pi G} \underline{\nabla} \times (\underline{v} \times \underline{g})} \quad -(26)$$

is the current of mass density.

Units The units of $\underline{J}_m$ are kgm $m^{-2} s^{-1}$, the units of ρ_m are kgm m^{-3}. The units of G are $m^3 s^{-2}$ kgm^{-1} and the units of $\underline{\Omega}$ are rad s^{-1}.

So:

$$\underline{J}_m = -\frac{1}{4\pi G} \left(\underline{v} (\underline{\nabla} \cdot \underline{g}) - (\underline{\nabla} \cdot \underline{v}) \underline{g} + (\underline{g} \cdot \underline{\nabla}) \underline{v} - (\underline{v} \cdot \underline{\nabla}) \underline{g} \right) \quad -(27)$$

and

$$\underline{\Omega} = -\frac{1}{c^2} \underline{v} \times \underline{g} \quad -(28)$$

5)

Now use:

$$\Omega^2 = \frac{1}{c^4}\,\underline{v}\times\underline{g}\cdot\underline{v}\times\underline{g} \quad -(29)$$

$$\boxed{\Omega^2 = \frac{1}{c^4}\left(v^2g^2 - (\underline{v}\cdot\underline{g})^2\right)} \quad -(29)a)$$

Eqs. (28) and (29) can be used with any type of orbital theory, and $\underline{J_n}$ worked out for any orbit.

<u>Example</u>

If $$\underline{g} = -\frac{MG}{r^2}\,\underline{e}_r \quad -(30)$$

then $$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(31)$$

and $$\underline{v} = \frac{dr}{dt}\,\underline{e}_r + \omega r\,\underline{e}_\theta. \quad -(32)$$

with $$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(33)$$

$$= MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(34)$$

where $$a = \frac{\alpha}{1-\epsilon^2} \quad -(35)$$

b) is the half major axis.

Therefore $\underline{\Omega}$, Ω^2 and $\underline{J}_m$ can be worked out for the conic section orbit (31).

Finally in ECE2:

$$\underline{\nabla} \times \underline{\Omega} = \underline{\kappa} \times \underline{\Omega} = \frac{4\pi G}{c^2} \underline{J}_m \qquad -(36)$$

where

$$\underline{\kappa} = \frac{1}{r^{(0)}} \underline{q} - \underline{\omega}_s \qquad -(37)$$

Here $\underline{q}$ is the tetrad vector and $\underline{\omega}_s$ the spin connection vector.

Conclusion

In general, $\underline{\Omega}$ is given by eq. (22), and for the centripetal acceleration $-\underline{\omega} \times (\underline{\omega} \times \underline{r})$, $\underline{\Omega}$ is given by Eq. (6).