Note 320(6): Complete Calculation of the Gravito-Magnetic Field.

The calculation begins with the general coordinate transform:

$$G^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} G^{\mu\nu} \quad -(1)$$

Here $G^{\mu\nu}$ is the rest frame of the gravitomagnetic field tensor. Here $G^{\mu\nu}$ is the rest frame, or frame in which the particle is at rest, and $G^{\mu'\nu'}$ the frame in which the particle moves with a velocity $\underline{v}$. This is the observer frame. For example, the sun, the earth if the observer is situated on the earth orbits the sun with velocity $\underline{v}$.

Eq. (1) means that:

$$\underline{g}' = \gamma(\underline{g} + \underline{v} \times \underline{\Omega}) - \frac{\gamma^2}{1+\gamma} \frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{g}\right) \quad -(2)$$

$$\underline{\Omega}' = \gamma\left(\underline{\Omega} - \frac{1}{c^2} \underline{v} \times \underline{g}\right) - \frac{\gamma^2}{1+\gamma} \frac{\underline{v}}{c}\left(\frac{\underline{v}}{c} \cdot \underline{\Omega}\right) \quad -(3)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

A Lorentz transform has been used because of the pseudo Lorentz transform property of the gravito-magnetic field equations.

2) In the limit:

$$v << c \quad -(5)$$

$$\underline{g}' = \underline{g} + \underline{v} \times \underline{\Omega} \quad -(6)$$

$$\underline{\Omega}' = \underline{\Omega} - \frac{1}{c^2} \underline{v} \times \underline{g} \quad -(7)$$

The analogous equations in electrodynamics are:

$$\underline{E}' = \underline{E} + \underline{v} \times \underline{B} \quad -(8)$$

$$\underline{B}' = \underline{B} - \frac{1}{c^2} \underline{v} \times \underline{E} \quad -(9)$$

In cylindrical polar coordinates the linear velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(10)$$

and the linear acceleration is:

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(11)$$

In Cartesian coordinates:

$$\underline{v} = \dot{r}\,\underline{e}_r \quad -(12)$$

and

$$\underline{a} = \ddot{r}\,\underline{e}_r \quad -(13)$$

In cylindrical polar coordinates <u>the frame is rotating</u>, and generates the extra terms.

3) In vector notation:

$$\underline{v} = \dot{r}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(14)$$

and:

$$\underline{a} = \ddot{r}\,\underline{e}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r \quad -(15)$$

where

$$\underline{e}_r = \underline{e}_\theta \times \underline{k} \quad -(16)$$

$$\underline{e}_\theta = \underline{k} \times \underline{e}_r \quad -(17)$$

$$\underline{k} = \underline{e}_r \times \underline{e}_\theta \quad -(18)$$

In gravitational theory,

$$\underline{a} = \underline{g} \quad -(19)$$

so

$$\underline{g} = \ddot{r}\,\underline{e}_r - r\dot{\theta}^2\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(20)$$

In this notation, eq. (6) is:

$$\underline{g} = \ddot{r}\,\underline{e}_r + \underline{v} \times \underline{\Omega} \quad -(21)$$

where

$$\underline{v} = \dot{r}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(22)$$

therefore in general:

$$\underline{v} \times \underline{\Omega} = -r\dot{\theta}^2\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(23)$$

4) <u>Planar Orbital Theory</u>

In this case:

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta = \underline{0} \quad -(24)$$

for any planar orbit. The result (24) follows from Lagrangian dynamics as in previous work. So for planar orbits:

$$\underline{v} = \dot{r}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(25)$$

and

$$\underline{a} = \ddot{r}\,\underline{e}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(26)$$

From eqs. (21) and (26):

$$\underline{v} \times \underline{\Omega} = -\underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(27)$$

The solution of eq. (27) is obtained from:

$$(\dot{r}\,\underline{e}_r + \underline{\omega} \times \underline{r}) \times \underline{\Omega} = -\underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(28)$$

The solution of eq. (28) is:

$$(\underline{\omega} \times \underline{r}) \times \underline{\Omega} = -\underline{\Omega} \times (\underline{\omega} \times \underline{r}) \quad -(29)$$
$$= -\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

i.e.

$$\underline{\Omega} = \underline{\omega} \quad -(30)$$

and

$$\underline{v} = \underline{\omega} \times \underline{r} \quad -(31)$$

Therefore in planar orbital theory the

5) gravitomagnetic field is:

$$\underline{\Omega} = \underline{\omega} = \frac{d\theta}{dt} \underline{k} \quad -(32)$$

which is the angular velocity of the rotating frame. For example $\underline{\Omega}$ is the angular velocity of the earth rotating about its axis. The velocity $\underline{v}$ of the Lorentz transform is the orbital linear velocity generated by the angular velocity of the rotating frame. This is a simple but profound result, indicating that planar orbital dynamics are the result of the ECE2 gravitomagnetic field theory, a generally covariant unified field theory. This is a natural way of describing orbital dynamics with general relativity.

In general in dynamics the Coriolis force is non-zero, so the gravitomagnetic field is in general:

$$\underline{\Omega} = \underline{\Omega}_1 . \quad -(33)$$

Now assume that:

$$\underline{v} \times \underline{\Omega}_1 = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) \underline{e}_\theta \quad -(34)$$

The solution is

$$\underline{v} = \frac{dr}{dt} \underline{e}_r \quad -(35)$$

and

$$\frac{dr}{dt} \underline{e}_r \times \underline{\Omega}_1 = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) \underline{e}_\theta \quad -(36)$$

6) Using:

$$\underline{e}_\theta = \underline{k} \times \underline{e}_r \quad -(37)$$

it follows that:

$$\underline{\Omega}_1 = -\frac{1}{\dot{r}}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{k} = -\left(\frac{r}{\dot{r}}\ddot{\theta} + 2\dot{\theta}\right)\underline{k} \quad -(38)$$

and:

$$\underline{a} = \ddot{r}\,\underline{e}_r + (\underline{\omega} \times \underline{r}) \times \underline{\Omega}_0 + \frac{dr}{dt}\,\underline{e}_r \times \underline{\Omega}_1 \quad -(39)$$

where:

$$\underline{\Omega}_0 = \omega\,\underline{k} \quad -(40)$$

and

$$\underline{\Omega}_1 = -\left(\frac{r}{\dot{r}}\ddot{\theta} + 2\dot{\theta}\right)\underline{k} \quad -(41)$$

Therefore in general, electrodynamics are described by gravitomagnetic Lorentz force equations.