320(3): Development of Eqs. (18) to (23) of Note 320(2)

The starting equation is:

$$\underline{v} \times \underline{\Omega} = \omega^2 r \, \underline{e}_r \quad -(1)$$

where

$$\underline{v} = \frac{dr}{dt} \underline{e}_r + \omega r \, \underline{e}_\theta \quad -(2)$$

Therefore:

$$\left(\frac{dr}{dt} \underline{e}_r + \omega r \, \underline{e}_\theta \right) \times \underline{\Omega} = \omega^2 r \, \underline{e}_r . \quad -(3)$$

Let

$$\underline{\Omega} = A \underline{e}_r + B \underline{e}_\theta + C \underline{k} \quad -(4)$$

in general. So:

$$\left(\frac{dr}{dt} \underline{e}_r + \omega r \, \underline{e}_\theta \right) \times \left(A \underline{e}_r + B \underline{e}_\theta + C \underline{k} \right) = \omega^2 r \, \underline{e}_r \quad -(5)$$

The unit vectors have an O(3) symmetry:

$$\underline{e}_r = \underline{e}_\theta \times \underline{k} \quad -(6)$$

$$\underline{e}_\theta = \underline{k} \times \underline{e}_r \quad -(7)$$

$$\underline{k} = \underline{e}_r \times \underline{e}_\theta \quad -(8)$$

The right hand side of eq. (5) only has an $\underline{e}_r$ component. There are six possible cross products on the left hand side of eq. (5):

2)

$$\left.\begin{aligned}
\underline{e}_r \times \underline{e}_r &= \underline{0} \\
\underline{e}_r \times \underline{e}_\theta &= \underline{k} \\
\underline{e}_r \times \underline{k} &= -\underline{e}_\theta \\
\underline{e}_\theta \times \underline{e}_r &= -\underline{k} \\
\underline{e}_\theta \times \underline{e}_\theta &= \underline{0} \\
\underline{e}_\theta \times \underline{k} &= \underline{e}_r
\end{aligned}\right] \quad -(9)$$

It follows that:

$$\underline{\Omega} = \Omega \underline{k} \quad -(10)$$

and

$$\omega r \underline{e}_\theta \times \Omega \underline{k} = \omega^2 r \underline{e}_r \quad -(11)$$

$$= \omega r \Omega \underline{e}_r$$

So

$$\boxed{\Omega = \omega} \quad -(12)$$

Q.E.D.