316(3): Details of the Procedure of Removing the Tangent Index.

First consider the two dimensional vector in the Cartesian and complex circular basis:

$$\underline{A} = A_x \underline{i} + A_y \underline{j} = A^{(1)} \underline{e}^{(1)} + A^{(2)} \underline{e}^{(2)} \quad -(1)$$

where

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) \quad -(2)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) \quad -(3)$$

It follows that:

$$A_x \underline{i} + A_y \underline{j} = \frac{1}{\sqrt{2}} \left(A^{(1)} (\underline{i} - i\underline{j}) + A^{(2)} (\underline{i} + i\underline{j}) \right) \quad -(4)$$

$$= \frac{1}{\sqrt{2}} \left(A^{(1)} + A^{(2)} \right) \underline{i} + \frac{i}{\sqrt{2}} \left(A^{(2)} - A^{(1)} \right) \underline{j}$$

So

$$A_x = \frac{1}{\sqrt{2}} \left(A^{(1)} + A^{(2)} \right) \quad -(5)$$

$$A_y = \frac{i}{\sqrt{2}} \left(A^{(2)} - A^{(1)} \right) \quad -(6)$$

and

$$A^{(1)} = \frac{1}{\sqrt{2}} (A_x + iA_y) \quad -(7)$$

$$A^{(2)} = \frac{1}{\sqrt{2}} (A_x - iA_y) \quad -(8)$$

2)

Now consider the vector:

$$\underline{A}^{(1)} = A_x^{(1)} \underline{i} + A_y^{(1)} \underline{j} \quad -(9)$$

and its complex conjugate:

$$\underline{A}^{(2)} = A_x^{(2)} \underline{i} + A_y^{(2)} \underline{j} \quad -(10)$$

and the Cartesian unit vector in 2-D

$$\underline{e} = 1 \underline{i} + 1 \underline{j} \quad -(11)$$

Now consider the tensorial equation:

$$A^a = A^a_\mu e^\mu \quad -(12)$$

where the first ECE hypothesis gives:

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(13)$$

where q^a_μ is the Cartan tetrad defined by:

$$V^a = q^a_\mu V^\mu \quad -(14)$$

where V^a is a vector in the tangent space and V^μ is the same vector in the base manifold. It follows that:

$$A^a = q^a_\mu A^\mu \quad -(15)$$

and

$$A^{(0)} A^a = A^a_\mu A^\mu \quad -(16)$$

3)

So
$$\boxed{A^a = A^a_\mu \left(\frac{A^\mu}{A^{(0)}} \right)} \quad -(17)$$

Denote
$$\epsilon^\mu = \frac{A^\mu}{A^{(0)}} \quad -(18)$$

then:
$$\boxed{A^a = A^a_\mu \, \epsilon^\mu} \quad -(19)$$

Similarly:
$$\boxed{A_\mu = A^a_\mu \, \epsilon_a} \quad -(20)$$

In two dimensions eq. (19) gives:
$$A^{(1)} = A^{(1)}_1 \epsilon^1 + A^{(1)}_2 \epsilon^2 \quad -(21)$$
and
$$A^{(2)} = A^{(2)}_1 \epsilon^1 + A^{(2)}_2 \epsilon^2 . \quad -(22)$$

Eqns. (21) and (22) must be the same as eqns. (7) and (8). Due to the minus sign introduced by contravariant covariant notation:
$$A^{(1)}_1 = - A^{(1)}_X \quad -(23)$$
and so on. So it is necessary to define eqs. (19) and (20) with a minus sign:

4)

$$A^a = -A^a_\mu \epsilon^\mu \quad -(24)$$

and

$$A_\mu = -A^a_\mu \epsilon_a . \quad -(25)$$

It follows that:

$$A^{(1)} = A^{(1)}_x \epsilon_x + A^{(1)}_y \epsilon_y \quad -(26)$$

and

$$A^{(2)} = A^{(2)}_x \epsilon_x + A^{(2)}_y \epsilon_y \quad -(27)$$

Comparing eqs. (7) and (26):

$$A^{(1)}_x \epsilon_x = \frac{A_x}{\sqrt{2}} \quad -(28)$$

and

$$A^{(1)}_y \epsilon_y = i\frac{A_y}{\sqrt{2}} . \quad -(29)$$

Comparing eqs. (8) and (27):

$$A^{(2)}_x \epsilon_x = \frac{A_x}{\sqrt{2}} \quad -(30)$$

$$A^{(2)}_y \epsilon_y = -i\frac{A_y}{\sqrt{2}} \quad -(31)$$

In these equations:

$$\epsilon_x = \frac{A_x}{A^{(0)}}, \quad \epsilon_y = \frac{A_y}{A^{(0)}} \quad -(32)$$

It follows that:

5)

$$A^{(1)}_x = \frac{A^{(0)}}{\sqrt{2}}, \quad A^{(1)}_y = -i\frac{A^{(0)}}{\sqrt{2}}, \quad -(33)$$

$$A^{(2)}_y = \frac{A^{(0)}}{\sqrt{2}}, \quad A^{(2)}_y = -i\frac{A^{(0)}}{\sqrt{2}} \quad -(34)$$

So

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad -(35)$$

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad -(36)$$

which is the self consistent and correct result, Q. E. D.

Similarly, from eq. (25):

$$A_1 = -\left(A^{(1)}_1 \epsilon_{(1)} + A^{(2)}_1 \epsilon_{(2)}\right) \quad -(37)$$

where

$$A^{(1)}_1 = -A^{(1)}_x = -\frac{A^{(0)}}{\sqrt{2}} \quad -(38)$$

and

$$A^{(2)}_1 = -A^{(2)}_x = -\frac{A^{(0)}}{\sqrt{2}}, \quad -(39)$$

Furthermore:

$$\epsilon_{(1)} = \frac{A_{(1)}}{A^{(0)}} = -\frac{1}{\sqrt{2}}(1+i) \quad -(40)$$

and

$$\epsilon_{(2)} = -\frac{1}{\sqrt{2}}(1-i) \quad -(41)$$

i.e.
$$\epsilon_{(1)} = -\frac{1}{\sqrt{2}\,A^{(0)}}\left(A_X + iA_Y\right) \quad -(42)$$

and
$$\epsilon_{(2)} = -\frac{1}{\sqrt{2}\,A^{(0)}}\left(A_X - iA_Y\right) \quad -(43)$$

So:
$$A_1 = -\frac{1}{2}\left(A_X + iA_Y + A_X - iA_Y\right) \quad -(44)$$
$$= -A_X$$

which is the correct result, Q.E.D.

Conclusion

The two index potential A^a_μ of ECE theory can be reduced to a one index potential using eqs. (24) and (25). This note checks the assumption made in Notes 316(1) and 316(2)