314(3): Development of the Evans Identity

From the previous notes the first Evans identity is:

$$F^{b}_{\mu a} \tilde{F}^{a\mu\nu} = 0 \quad -(1)$$

where

$$F^{b}_{\mu a} = F^{b}_{\mu\nu} q^{\nu}_{a} \quad -(2)$$

so

$$F^{b}_{\mu\nu} = q^{a}_{\nu} F^{b}_{\mu a} \quad -(3)$$

i.e.

$$\boxed{F^{b}_{\mu\nu} = A^{a}_{\nu} T^{b}_{\mu a}} \quad -(4)$$

which is a new structure equation of differential geometry, i.e.: $\boxed{T^{b}_{\mu\nu} = q^{a}_{\nu} T^{b}_{\mu a}}$, $-(5)$

a relation between the torsion and tetrad. Eq. (5) is therefore equivalent to the first Cartan structure equation:

$$T = D \wedge q. \quad -(6)$$

The tensor valued one form $T^{b}_{\mu a}$ plays the role of the covariant wedge derivative $D\wedge$.

From previous notes eq. (1) gives:

$$\underline{E}^{b} \cdot \underline{B}^{a} + \underline{B}^{b} \cdot \underline{E}^{a} = 0 \quad -(7)$$

and also:

2)

$$\underline{F}^{b}{}_{a} \cdot \underline{B}^{a} = 0 \quad -(8)$$

$$c F^{b}{}_{0a} \underline{B}^{a} = \underline{F}^{b}{}_{a} \times \underline{E}^{a} \quad -(9)$$

For $b = (1)$, eq. (8) is:

$$\underline{F}^{(1)}{}_{(2)} \cdot \underline{B}^{(2)} + \underline{F}^{(1)}{}_{(1)} \cdot \underline{B}^{(1)} = 0 \quad -(10)$$

and for $b = (2)$, eq. (8) is:

$$\underline{F}^{(2)}{}_{(1)} \cdot \underline{B}^{(1)} + \underline{F}^{(2)}{}_{(2)} \cdot \underline{B}^{(2)} = 0 \quad -(11)$$

Using these polarizations eq. (7) becomes:

$$\underline{E}^{(1)} \cdot \underline{B}^{(2)} + \underline{E}^{(2)} \cdot \underline{B}^{(1)} = 0 \quad -(12)$$

Therefore:

$$\boxed{\begin{aligned} \underline{E}^{(1)} &= \underline{F}^{(2)}{}_{(2)} = \underline{F}^{(1)}{}_{(2)} \quad -(13) \\ \underline{E}^{(2)} &= \underline{F}^{(2)}{}_{(1)} = \underline{F}^{(1)}{}_{(1)} \quad -(14) \end{aligned}}$$

and

Eq. (9) is:

$$c\left(F^{b}{}_{0(1)} \underline{B}^{(1)} + F^{b}{}_{0(2)} \underline{B}^{(2)}\right) = \underline{F}^{b}{}_{(1)} \times \underline{E}^{(1)} + \underline{F}^{b}{}_{(2)} \times \underline{E}^{(2)} \quad -(15)$$

If:

$$b = (2) \quad -(16)$$

then

3)

$$c\left(F^{(2)}_{0(1)}\underline{B}^{(1)} + F^{(2)}_{0(2)}\underline{B}^{(2)}\right) = \underline{F}^{(2)}_{(1)} \times \underline{E}^{(1)} + \underline{F}^{(2)}_{(2)} \times \underline{E}^{(2)}$$

$$= \underline{E}^{(2)} \times \underline{E}^{(1)} + \underline{E}^{(1)} \times \underline{E}^{(2)} \quad - (16)$$

$$= \underline{0}$$

So:

$$\boxed{F^{(2)}_{0(1)}\underline{B}^{(1)} + F^{(2)}_{0(2)}\underline{B}^{(2)} = \underline{0}} \quad - (17)$$

If

$$b = (1) \quad - (18)$$

$$c\left(F^{(1)}_{0(1)}\underline{B}^{(1)} + F^{(1)}_{0(2)}\underline{B}^{(2)}\right) = \left(\underline{F}^{(1)}_{(1)} \times \underline{E}^{(1)} + \underline{F}^{(1)}_{(2)} \times \underline{E}^{(2)}\right)$$

$$= \underline{E}^{(2)} \times \underline{E}^{(1)} + \underline{E}^{(1)} \times \underline{E}^{(2)} \quad - (19)$$

$$= \underline{0}$$

So

$$\boxed{F^{(1)}_{0(1)}\underline{B}^{(1)} + F^{(1)}_{0(2)}\underline{B}^{(2)} = \underline{0}} \quad - (20)$$

The units of the time like $F_0{}^{(2)}{}_{(1)}$ etc. are those of electric field strength $\underline{E}$.

Another possible solution is:

$$\underline{E}^{(1)} \times \underline{E}^{(2)} = c F_0{}^{(1)}{}_{(2)} \underline{B}^{(2)} \quad - (21)$$

but this contradicts the fact that:

$$\underline{E}^{(1)} \times \underline{E}^{(2)} = -ic^2 B^{(0)} \underline{B}^{(3)*} \quad -(22)$$

Therefore eqs. (17) and (20) are the only possible solutions.

One possibility is that:

$$F^{(2)}{}_{0(1)} = F^{(2)}{}_{0(2)} = F^{(1)}{}_{0(1)} = F^{(1)}{}_{0(2)} = 0 \quad -(23)$$

So eq. (9) reduces to:

$$\underline{F}^{b}{}_{a} \times \underline{E}^{a} = \underline{0} \quad -(24)$$

Eq. (23) could be interpreted to mean that there are no time like components of an electric field. We therefore arrive at:

$$\boxed{\begin{aligned} \underline{E}^{b} \cdot \underline{B}^{a} + \underline{B}^{b} \cdot \underline{E}^{a} &= 0 \\ \underline{F}^{b}{}_{a} \cdot \underline{B}^{a} &= 0 \\ \underline{F}^{b}{}_{a} \times \underline{E}^{a} &= \underline{0} \end{aligned}} \quad -(25)$$

These are relations between free space fields. The next notes will deal with other types of field.