314(5): Proof of Some Tensor Algebra

In Note 314(2) it was shown that the first Evans identity is:

$$T^b{}_{\mu a}\tilde{T}^{a\mu\nu} = 0 \quad -(1)$$

where

$$T^b{}_{\mu a} = q^\nu{}_a T^b{}_{\mu\nu}\tilde{T}^{a\mu\nu} \quad -(2)$$

We wish to prove that:

$$T^b{}_{\mu\nu}\tilde{T}^{a\mu\nu} = 0 \quad -(3)$$

is a possible solution of eqs. (1) and (2).

Proof Consider the equation:

$$V_a V^\mu = 0 \quad -(4)$$

In this equation: $V_a = q^\mu{}_a V_\mu \quad -(5)$

so

$$q^\mu{}_a V_\mu V^\mu = 0 \quad -(6)$$

In order to explain the meaning of the summation in eq. (6) assume that:

$$(q^\mu{}_a V_\mu) V^\mu = q^\mu{}_a (V_\mu V^\mu) = 0 \quad -(7)$$

where summation over repeated indices occurs inside the brackets.

d) So:

$$q^{\mu}_{a}\left(V_0 V^0 + V_1 V^1 + V_2 V^2 + V_3 V^3\right)$$
$$= \left(q^0_a V_0 + q^1_a V_1 + q^2_a V_2 + q^3_a V_3\right) V^{\mu} = 0 \quad -(8)$$

For $\mu = 0$:

$$q^0_a\left(V_0 V^0 + V_1 V^1 + V_2 V^2 + V_3 V^3\right) = 0$$
$$= \left(q^0_a V_0 + q^1_a V_1 + q^2_a V_2 + q^3_a V_3\right) V^0 \quad -(9)$$

with similar results for

$\mu = 1, 2, 3 \quad -(10)$

Adding:

$$\left(q^0_a + q^1_a + q^2_a + q^3_a\right)\left(V_0 V^0 + V_1 V^1 + V_2 V^2 + V_3 V^3\right)$$
$$\left(V^0 + V^1 + V^2 + V^3\right)\left(q^0_a V_0 + q^1_a V_1 + q^2_a V_2 + q^3_a V_3\right)$$
$$= 0 \quad -(11)$$

Eq. (11) is algebraically correct. It can be expressed as:

$$\sum_{\mu=0}^{3} V^{\mu}\left(q^{\nu}_a V_{\nu}\right) = \sum_{\mu=0}^{3} q^{\mu}_a\left(V^{\nu} V_{\nu}\right)$$
$$= 0 \quad -(12)$$

3) in which the summation indices inside the brackets have been changed from μ to ν.

Similarly, eq. (7) can be expressed as:

$$\left(q^{\nu}{}_{a} \nabla_{\nu}\right) \nabla^{\mu} = q^{\mu}{}_{a} \left(\nabla_{\nu} \nabla^{\nu}\right) \quad -(13)$$

Eq. (12) is always true algebraically, and eq. (13) is a particular solution of eq. (12) in which all four components of each eq. (12) are zero. Eq. (12) can always be written as: $= 0$

$$\sum_{\mu=0}^{3} \nabla^{\mu} \left(q^{\mu}{}_{a} \nabla_{\mu}\right) = \sum_{\mu=0}^{3} q^{\mu}{}_{a} \left(\nabla^{\mu} \nabla_{\mu}\right) \quad -(14)$$

but it is less confusing to write it as eq. (12), in which it is clear that the outer summation is over μ, and that there is another summation over ν.

It is clear that a possible solution of eq. (12) is:

$$\nabla^{\nu} \nabla_{\nu} = 0 \quad -(15)$$

Similarly, the Evans identity (2) is:

$$q^{\nu}{}_{a} T^{b}{}_{\mu\nu} \tilde{T}^{a\mu\nu} = 0 \quad -(16)$$

4) The structure of eq. (16) is directly analogous to:

$$\nabla^{\sim}_a V_{\sim} V^{\sim} = 0 \quad -(17)$$

By direct analogy with eq. (12) the following equation is always true:

$$\sum_{\mu=0}^{3} \nabla^{\mu}_{a}\left(T^{b}_{\rho\nu}\tilde{T}^{a\rho\nu}\right) = \sum_{\mu=0}^{3}\tilde{T}^{a\mu\nu}\left(\nabla^{\rho}_{a}T^{b}_{\rho\nu}\right)$$

$$= 0 \quad -(18)$$

It is clear that a possible solution of eq. (18) is:

$$T^{b}_{\rho\nu}\tilde{T}^{a\rho\nu} = 0 \quad -(19)$$

which is eq. (3), Q.E.D.