313(4): Inference of New Bianchi Identities with Torsion: ReBCE and JCE Identities

Consider eqs. (15) and (16) of Note 313(2). It is assumed that Eq. (16) is a solution of Eq. (15), which is obtained directly from the Cartan identity.

Proof Add Eqs. (15) and (16) to give:

$$
\begin{aligned}
&D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} = D_\mu D_\lambda T^\kappa{}_{\nu\rho} + D_\rho D_\lambda T^\kappa{}_{\mu\nu} + D_\nu D_\lambda T^\kappa{}_{\rho\mu} \\
&+ D_\mu \left(R^\kappa{}_{\rho\lambda\nu} + R^\kappa{}_{\nu\rho\lambda} + R^\kappa{}_{\lambda\nu\rho} \right) \qquad + D_\mu \left(D_\rho T^\kappa{}_{\lambda\nu} + D_\nu T^\kappa{}_{\rho\lambda} + D_\lambda T^\kappa{}_{\nu\rho} \right) \\
&+ D_\rho \left(R^\kappa{}_{\nu\lambda\mu} + R^\kappa{}_{\mu\nu\lambda} + R^\kappa{}_{\lambda\mu\nu} \right) \qquad + D_\rho \left(D_\nu T^\kappa{}_{\lambda\mu} + D_\mu T^\kappa{}_{\nu\lambda} + D_\lambda T^\kappa{}_{\mu\nu} \right) \\
&+ D_\nu \left(R^\kappa{}_{\mu\lambda\rho} + R^\kappa{}_{\rho\mu\lambda} + R^\kappa{}_{\lambda\rho\mu} \right) \qquad + D_\nu \left(D_\mu T^\kappa{}_{\lambda\rho} + D_\rho T^\kappa{}_{\mu\lambda} + D_\lambda T^\kappa{}_{\rho\mu} \right)
\end{aligned}
\quad -(1)
$$

However, it is known from the Cartan identity that:

$$
R^\kappa{}_{\rho\lambda\nu} + R^\kappa{}_{\nu\rho\lambda} + R^\kappa{}_{\lambda\nu\rho} := D_\rho T^\kappa{}_{\lambda\nu} + D_\nu T^\kappa{}_{\rho\lambda} + D_\lambda T^\kappa{}_{\nu\rho} \quad -(2)
$$

and so on. It follows that:

$$
\begin{aligned}
&D_\mu D_\lambda T^\kappa{}_{\nu\rho} + D_\rho D_\lambda T^\kappa{}_{\mu\nu} + D_\nu D_\lambda T^\kappa{}_{\rho\mu} \\
&\quad := D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu}
\end{aligned}
\quad -(3)
$$

Q.E.D.

2) Eq. (3) is an exact identity and was first given in UFT 255, Eq. (105), reviewed in UFT 281. The second Bianchi identity of 1902 is:

$$D_\mu R^{\kappa}{}_{\lambda\nu\rho} + D_\rho R^{\kappa}{}_{\lambda\mu\nu} + D_\nu R^{\kappa}{}_{\lambda\rho\mu} = 0 \quad -(4)$$

i.e. Bianchi did not know of the existence of torsion.

In an exactly similar manner to the above proof it can be shown that there exist two new identities:

$$D_\mu R^{\kappa}{}_{\rho\lambda\nu} + D_\rho R^{\kappa}{}_{\nu\lambda\mu} + D_\nu R^{\kappa}{}_{\mu\lambda\rho} := D_\mu D_\rho T^{\kappa}{}_{\mu\nu} + D_\rho D_\nu T^{\kappa}{}_{\lambda\mu} + D_\nu D_\mu T^{\kappa}{}_{\lambda\rho} \quad -(5)$$

and

$$D_\mu R^{\kappa}{}_{\nu\rho\lambda} + D_\rho R^{\kappa}{}_{\mu\nu\lambda} + D_\nu R^{\kappa}{}_{\rho\mu\lambda} := D_\mu D_\nu T^{\kappa}{}_{\rho\lambda} + D_\rho D_\mu T^{\kappa}{}_{\nu\lambda} + D_\nu D_\rho T^{\kappa}{}_{\mu\lambda} \quad -(6)$$

It can be seen that the index λ moves across to the left from eq. (3) to eq. (6). The upper index remains κ. The cyclic permutation of μ, ν and ρ remains

3) In addition to these new identities, which are named the Bianchi Cartan Evans identities (BCE Identities) here are the new identities of Note 313(3) inferred from the Jacobi identity:

$$\left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho}\right) V^\lambda := \left(D_\rho T^\lambda{}_{\mu\nu} + D_\nu T^\lambda{}_{\rho\mu} + D_\mu T^\lambda{}_{\nu\rho}\right) D_\lambda V^\kappa, \quad -(7)$$

$$\left(R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\nu\rho\mu} + R^\lambda{}_{\mu\nu\rho}\right) D_\lambda V^\kappa := \left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho}\right) V^\lambda \quad -(8)$$

and:

$$\left(D_\rho T^\lambda{}_{\mu\nu} + D_\nu T^\lambda{}_{\rho\mu} + D_\mu T^\lambda{}_{\nu\rho}\right) D_\lambda V^\kappa := \left(D_\rho D_\lambda T^\kappa{}_{\mu\nu} + D_\nu D_\lambda T^\kappa{}_{\rho\mu} + D_\mu D_\lambda T^\kappa{}_{\nu\rho}\right) V^\lambda \quad -(9)$$

These are named the Jacobi Cartan Evans identities (JCE identities).

More generally eqs. (7) to (9) can be extended to any tensor, upon which

4) acts the commutator of covariant derivatives.

The case relevant to electrodynamics and gravitation is when the commutator of covariant derivatives acts on the Cartan tetrad.

In the BCE and JCE identities the connection is always antisymmetric:

$$\Gamma^{\lambda}_{\mu\nu} = - \Gamma^{\lambda}_{\mu\nu} \quad - (10)$$

Note carefully that the use of the usual Christoffel connection:

$$\Gamma^{\lambda}_{\mu\nu} \overset{?}{=} \Gamma^{\lambda}_{\nu\mu} \quad - (11)$$

means that gravitation of the Einsteinian type vanishes as discussed in Note 313(3).