313(8): Checking and Clarifying Note 313(6)

From Eq. (14) onwards:

$$[D_\rho, [D_\mu, D_\nu]] V^\kappa = D_\rho \left([D_\mu, D_\nu] V^\kappa \right) - [D_\mu, D_\nu] D_\rho V^\kappa \quad -(1)$$

$$= D_\rho \left(R^\kappa{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} D_\lambda V^\kappa \right) - R^\kappa{}_{\lambda\mu\nu} D_\rho V^\lambda + R^\lambda{}_{\rho\mu\nu} D_\lambda V^\kappa + T^\lambda{}_{\mu\nu} D_\lambda D_\rho V^\kappa$$

$$= D_\rho R^\kappa{}_{\lambda\mu\nu} V^\lambda + R^\kappa{}_{\lambda\mu\nu} D_\rho V^\lambda - D_\rho T^\lambda{}_{\mu\nu} D_\lambda V^\kappa - T^\lambda{}_{\mu\nu} D_\rho D_\lambda V^\kappa - R^\kappa{}_{\lambda\mu\nu} D_\rho V^\lambda + R^\lambda{}_{\rho\mu\nu} D_\lambda V^\kappa + T^\lambda{}_{\mu\nu} D_\lambda D_\rho V^\kappa$$

$$= D_\rho R^\kappa{}_{\lambda\mu\nu} V^\lambda - D_\rho T^\lambda{}_{\mu\nu} D_\lambda V^\kappa + R^\lambda{}_{\rho\mu\nu} D_\lambda V^\kappa - T^\lambda{}_{\mu\nu} [D_\rho, D_\lambda] V^\kappa$$

The complete Jacobi identity is therefore:

$$\left([D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] + [D_\mu, [D_\nu, D_\rho]] \right) V^\kappa$$
$$= \left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho} \right) V^\lambda$$

2)

$$+\left(R^{\lambda}_{\ \rho\mu\nu}+R^{\lambda}_{\ \nu\rho\mu}+R^{\lambda}_{\ \mu\nu\rho}-\left(D_\rho T^{\lambda}_{\ \mu\nu}+D_\nu T^{\lambda}_{\ \rho\mu}+D_\mu T^{\lambda}_{\ \nu\rho}\right)\right)D_\lambda V^{\kappa}$$

$$-\left(T^{\lambda}_{\ \mu\nu}[D_\rho, D_\lambda]+T^{\lambda}_{\ \rho\mu}[D_\nu, D_\lambda]+T^{\lambda}_{\ \nu\rho}[D_\mu, D_\lambda]\right)V^{\kappa}$$

$$:= 0 \qquad -(2)$$

Now use the Cartan identity:

$$D_\rho T^{\lambda}_{\ \mu\nu}+D_\nu T^{\lambda}_{\ \rho\mu}+D_\mu T^{\lambda}_{\ \nu\rho} := R^{\lambda}_{\ \rho\mu\nu}+R^{\lambda}_{\ \nu\rho\mu}+R^{\lambda}_{\ \mu\nu\rho} \qquad -(3)$$

<u>The</u> <u>Jacobi</u> <u>Cartan</u> <u>Evans</u> (<u>JCE</u>) <u>identity</u> is:

$$\left([D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]\right)V^{\kappa}$$

$$:=\left(T^{\lambda}_{\ \mu\nu}[D_\rho, D_\lambda]+T^{\lambda}_{\ \rho\mu}[D_\nu, D_\lambda]+T^{\lambda}_{\ \nu\rho}[D_\mu, D_\lambda]\right)V^{\kappa}$$

$$+\left(D_\rho R^{\kappa}_{\ \lambda\mu\nu}+D_\nu R^{\kappa}_{\ \lambda\rho\mu}+D_\mu R^{\kappa}_{\ \lambda\nu\rho}\right)V^{\lambda} \qquad -(4)$$

In this identity:

$$\left(T^{\lambda}_{\ \mu\nu}[D_\rho, D_\lambda]+T^{\lambda}_{\ \rho\mu}[D_\nu, D_\lambda]+T^{\lambda}_{\ \nu\rho}[D_\mu, D_\lambda]\right)V^{\kappa}$$

3)

$$= \left(T^{\lambda}_{\mu\nu} R^{\kappa}{}_{\alpha\rho\lambda} + T^{\lambda}_{\rho\mu} R^{\kappa}{}_{\alpha\nu\lambda} + T^{\lambda}_{\nu\rho} R^{\kappa}{}_{\alpha\mu\lambda} \right) V^{\alpha}$$

$$- \left(T^{\lambda}_{\mu\nu} T^{\alpha}_{\rho\lambda} + T^{\lambda}_{\rho\mu} T^{\alpha}_{\nu\lambda} + T^{\lambda}_{\nu\rho} T^{\alpha}_{\mu\lambda} \right) D_{\alpha} V^{\kappa} \quad - (5)$$

with the first Evans identity:

$$T^{\lambda}_{\mu\nu} T^{\alpha}_{\rho\lambda} + T^{\lambda}_{\rho\mu} T^{\alpha}_{\nu\lambda} + T^{\lambda}_{\nu\rho} T^{\alpha}_{\mu\lambda} := 0 \quad - (6)$$

So the Jacobi Cartan Evans identity is:

$$\left([D_{\rho}, [D_{\mu}, D_{\nu}]] + [D_{\nu}, [D_{\rho}, D_{\mu}]] + [D_{\mu}, [D_{\nu}, D_{\rho}]] \right) V^{\kappa}$$

$$:= \left(D_{\rho} R^{\kappa}{}_{\lambda\mu\nu} + D_{\nu} R^{\kappa}{}_{\lambda\rho\mu} + D_{\mu} R^{\kappa}{}_{\lambda\nu\rho} \right) V^{\lambda}$$

$$+ \left(T^{\lambda}_{\mu\nu} R^{\kappa}{}_{\alpha\rho\lambda} + T^{\lambda}_{\rho\mu} R^{\kappa}{}_{\alpha\nu\lambda} + T^{\lambda}_{\nu\rho} R^{\kappa}{}_{\alpha\mu\lambda} \right) V^{\alpha}$$

$$:= \left(D_{\rho} R^{\kappa}{}_{\lambda\mu\nu} + D_{\nu} R^{\kappa}{}_{\lambda\rho\mu} + D_{\mu} R^{\kappa}{}_{\lambda\nu\rho} + T^{\alpha}_{\mu\nu} R^{\kappa}{}_{\lambda\rho\alpha} + T^{\alpha}_{\rho\mu} R^{\kappa}{}_{\lambda\nu\alpha} + T^{\alpha}_{\nu\rho} R^{\kappa}{}_{\lambda\mu\alpha} \right) V^{\lambda} \quad - (7)$$

4) The original 1902 Bianchi identity is:

$$D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho} = 0 \quad -(8)$$

and this is no longer true in the presence of torsion.

The Jacobi identity (2) gives both the Cartan identity (3) and the first Evans identity (6), as well as the JCE identity (7).