305(1): Effect of Gravitation on the Orbitals of the H Atom

In the non relativistic limit of the ECE equation the Schroedinger equation is obtained. The latter is the quantization of the hamiltonian:

$$H = E + V \quad -(1)$$

$$= \frac{p^2}{2m} + V$$

using:

$$\underline{p}\,\phi = -i\hbar\underline{\nabla}\phi \quad -(2)$$

so

$$H\phi = \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\phi = E\phi \quad -(3)$$

In the H atom the usual potential energy is the Coulombic attraction between the electron and the proton:

$$V = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(4)$$

where r is the distance between the electron and proton, $-e$ is the charge on the electron, and ϵ_0 the vacuum permittivity.

The solution of eq. (3) is:

$$\phi(r, \theta, \phi) = R(r)Y(\theta, \phi) \quad -(5)$$

where $Y(\theta, \phi)$ are the spherical harmonics and where $R(r)$ are the radial wavefunctions:

2)

$$R_{nl}(r) = -\left(\frac{2Z}{na}\right)\left[\frac{(n-l-1)!}{2n((n+l)!)^3}\right]\rho^{l} L_{n+l}^{2l+1}(\rho) e^{-\rho/2} \quad -(6)$$

where:

$$\rho = \frac{2Z}{na} \quad -(7)$$

where

$$a = \frac{4\pi\epsilon_0 \hbar^2}{\mu e^2} \quad -(8)$$

Here L are the associated Laguerre functions. For atomic H:

$$Z = 1 \quad -(9)$$

and

$$\mu \doteq m \quad -(10)$$

where m is the mass of the electron. Then a becomes the Bohr radius:

$$a = \frac{4\pi\epsilon_0 \hbar^2}{m e^2} \quad -(11)$$

The quantum numbers of the H atom are n, l and

$$m = -l, l-1, \ldots, l. \quad -(12)$$

In the presence of gravitation the potential energy is changed to:

$$V = -\frac{e^2}{4\pi\epsilon_0 r} - \frac{m_1 m_2 G}{r} \quad -(13)$$

where G is Newton's constant and where

3) m_1 and m_2 are two interacting masses separated by a distance r. For example m_1 is the mass of the electron and m_2 is the mass of the proton. In S.I. units:

$$e = 1.60219 \times 10^{-19}\ C$$
$$4\pi\epsilon_0 = 1.112650 \times 10^{-10}\ J^{-1}C^2m^{-1}$$
$$m_1 = 9.10953 \times 10^{-31}\ kg$$
$$m_2 = 1.67265 \times 10^{-27}\ kg$$
$$G = 6.6726 \times 10^{-11}\ Nm^2kg^{-2}$$

Therefore:

$$V = -\left(\frac{e^2}{4\pi\epsilon_0} + m_1 m_2 G\right) / r \quad - (14)$$

This equation gives the same atomic orbitals of H, but in the Bohr radius:

$$a = \frac{\hbar^2}{m}\left(\frac{e^2}{4\pi\epsilon_0} + m m_2 G\right)^{-1} \quad - (15)$$

where m_2 is the mass of the proton. It is seen that

$$\frac{e^2}{4\pi\epsilon_0} = 4.3598 \times 10^{-18}\ Jm \quad - (16)$$

at the Bohr radius $\quad r = a \quad - (17)$

taken as an example of r:

$$r = 5.29177 \times 10^{-11}\ m \quad - (18)$$

4) and

$$m_1 m_2 G = 9.10953 \times 1.67265 \times 6.6726 \times 10^{-69}$$

$$= 1.01671 \times 10^{-67} \text{ Jm} \quad -(14)$$

Therefore in the H atom, $e^2/(4\pi\epsilon_0)$ is forty nine orders of magnitude larger than $m_1 m_2 G$.

Therefore the gravitational effect of the proton on the electron is entirely negligible.

However, the effect of the earth's gravitation on the electron and proton must be also be considered. The proton is much heavier than the electron so by far the larger effect is that of the earth's gravitation on the proton:

$$V = -\frac{M m_2 G}{R} \quad -(15)$$

The mass of the earth is:

$$\underline{M} = 5.972 \times 10^{24} \text{ kg} \quad -(16)$$

and its radius is:

$$R = 6.371 \times 10^{6} \text{ m} \quad -(17)$$

so from eq. (15):

$$V = -\frac{5.972 \times 1.67265 \times 6.6726 \times 10^{-14}}{6.3671 \times 10^{6}}$$

$$= 1.00431 \times 10^{-19}\ \text{J} \quad -(18)$$

The energy of electrostatic interaction between the proton and electron is:

$$V = - \frac{e^2}{4\pi \epsilon_0 r}$$

$$= 4.3598 \times 10^{-17}\ \text{J} \quad -(19)$$

So the effect of the earth's gravity on the proton is a hundredth of that of the electron on the proton. This gravitational effect is easily measurable in high resolution spectroscopy. Therefore the complete potential that should be used in the Schrodinger equation is:

$$V = -\left(\frac{e^2}{4\pi \epsilon_0 r} + \frac{m_2 M G}{R} \right) \quad -(20)$$

Denote:

$$R = x r \quad -(21)$$

then

$$\boxed{V = -\left(\frac{e^2}{4\pi \epsilon_0} + \frac{m_2 M G}{x} \right) \Big/ r} \quad -(21)$$

where

$$x = \frac{6.6371 \times 10^{6}}{5.29177 \times 10^{-11}} \quad -(22)$$

$$= 1.2039 \times 10^{-17}$$

) The atomic orbitals of H in the earth's gravitational field must therefore be worked out with:

$$a = \frac{\hbar^2}{m}\left(\frac{e^2}{4\pi\epsilon_0} + \frac{M m_2 G}{x}\right)^{-1} \quad -(23)$$

i.e.

$$\boxed{a = \frac{\hbar^2}{m\left(\frac{e^2}{4\pi\epsilon_0} + \frac{m_2 M G}{x}\right)}} \quad -(24)$$

Therefore in H: $\quad -(25)$

$$\boxed{\rho = \frac{2}{na} = \frac{2}{nm\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0} + \frac{m_2 M G}{x}\right)}$$

The orbitals are all affected to a different extent by the effect of the earth's gravitation on the proton. For example:

1s Orbital

$$n = 1, \quad l = 0, \quad R_{10}(r) = \left(\frac{1}{a}\right)^{3/2} \exp\left(-\frac{\rho}{2}\right) \quad -(26)$$

and so on as follows.

7) 2s Orbital

$n = 2,\ l = 0,$

$$R_{20} = \left(\frac{1}{a}\right)^{3/2}\left(\frac{1}{2\sqrt{2}}\right)(2-\rho)\exp\left(-\frac{\rho}{2}\right) \quad -(27)$$

2p Orbital

$n = 2,\ l = 1$

$$R_{21} = \left(\frac{1}{a}\right)^{3/2}\left(\frac{1}{2\sqrt{6}}\right)\rho\exp\left(-\frac{\rho}{2}\right) \quad -(28)$$

3s Orbital

$n = 3,\ l = 0,$

$$R_{30} = \left(\frac{1}{a}\right)^{3/2}\left(\frac{1}{9\sqrt{3}}\right)\left(6-6\rho+\rho^2\right)\exp\left(-\frac{\rho}{2}\right) \quad -(29)$$

3p Orbital

$n = 3,\ l = 1,$

$$R_{31} = \left(\frac{1}{a}\right)^{3/2}\left(\frac{1}{9\sqrt{6}}\right)(4-\rho)\rho\exp\left(-\frac{\rho}{2}\right) \quad -(30)$$

3d Orbital

$$R_{32} = \left(\frac{1}{a}\right)^{3/2}\left(\frac{1}{9\sqrt{30}}\right)\rho^2\exp\left(-\frac{\rho}{2}\right) \quad -(31)$$

So the Balmer $n = 2$ to $n = 3$ line is affected by gravitation.