504(8): Photon Mass from Any Absorption

The basic method is to assume that the e/m phase is:

$$\phi = \omega t - \frac{\omega z}{v} \quad -(1)$$

where

$$\frac{1}{v} = \frac{1}{c}\left(n' - i n''\right) \quad -(2)$$

So v is a complex number:

$$v = v' + i v'' \quad -(3)$$

$$v^* = v' - i v'' \quad -(4)$$

The square modulus of v is:

$$|v|^2 = v'^2 + v''^2 = v v^* \quad -(5)$$

By complex algebra:

$$v = v' + i v'' = \frac{c}{n' - i n''} = \frac{c\left(n' + i n''\right)}{n'^2 + n''^2} \quad -(6)$$

So

$$\frac{v'}{c} = \frac{n'}{n'^2 + n''^2} \quad -(7)$$

and

$$\frac{v''}{c} = \frac{n''}{n'^2 + n''^2} \quad -(8)$$

where:

$$\alpha = \frac{\omega \epsilon''}{n' c}, \quad n'' = \frac{\alpha c}{2\omega} \quad -(9)$$

Similarly:

2)
$$\frac{1}{v' + iv''} = \frac{1}{c}\left(n' - in''\right) \quad - (10)$$

$$= \frac{v' - iv''}{v'^2 + v''^2}$$

So:
$$\frac{n'}{c} = \frac{v'}{v'^2 + v''^2}, \quad \frac{n''}{c} = \frac{v''}{v'^2 + v''^2} \quad - (11)$$

$$= \frac{v'}{v^2} \qquad\qquad = \frac{v''}{v^2}$$

From 304(7) checked by computer algebra:
$$\epsilon'' = \frac{\alpha c^2}{2\omega^2 v'}\left(\left(\omega^2 - \alpha^2 v'^2\right)^{1/2} + \omega\right) - (12)$$

Using:
$$\alpha = \frac{\omega \epsilon''}{n' c} \quad - (13)$$

it is found that: $\quad - (14)$

$$\boxed{n' = \frac{c}{2\omega v'}\left(\left(\omega^2 - \alpha^2 v'^2\right)^{1/2} + \omega\right)}$$

This is the correct expression for n' in the presence of absorption. Note that if:
$$\alpha \rightarrow 0 \quad - (15)$$

3) then:

$$n' \to \frac{c}{v'} \quad -(16)$$

which is the usual expression used in textbooks.

The power absorption coefficient integrated over a band is:

$$\alpha = \frac{d}{v} \quad -(17)$$

where:

$$d = \left(\frac{N}{V}\right)\frac{|\mu_{fi}|^2}{6\epsilon_0 \hbar} \quad -(18)$$

From eq. (11):

$$\frac{n'}{c} = \frac{v'}{v^2} \quad -(19)$$

and

$$\alpha^2 v'^2 = d^2\left(\frac{v'}{v}\right)^2 \quad -(20)$$

So eq. (14) is:

$$\left(\frac{v'}{v}\right)^2 = \frac{1}{2\omega}\left(\left(\omega^2 - d^2\left(\frac{v'}{v}\right)^2\right)^{1/2} + \omega\right) \quad -(21)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\omega}\left(\omega^2 - d^2\left(\frac{v'}{v}\right)^2\right)^{1/2}\right)$$

This can be solved by computer algebra for v'/v in terms of d.

From eq. (11):

4)

$$\frac{n''}{c} = \frac{v''}{v^2} = \frac{\alpha}{2\omega} = \frac{d}{2v\omega} \quad -(22)$$

Therefore:

$$\boxed{\frac{v''}{v} = \frac{\alpha V}{2\omega}} \quad -(23)$$

By definition:

$$\left(\frac{v'}{v}\right)^2 + \left(\frac{v''}{v}\right)^2 = 1 \quad -(24)$$

Fm eqs. (21), (23) and (24) αV can be found in terms of d and ω. Therefore v can be expressed in terms of α.

If

$$x := \left(\frac{v'}{v}\right)^2 \quad -(25)$$

then:

$$\left(\frac{\alpha V}{2\omega}\right)^2 + x^2 = 1 \quad -(26)$$

and

$$\boxed{\alpha^2 V^2 = 2\omega\left(1 - x^2\right)} \quad -(27)$$

This gives the required relation between α and V.

5) From eq (21):

$$x = \frac{1}{2}\left(1 + \frac{1}{\omega}\left(\omega^2 - d^2 x\right)^{1/2}\right) \quad -(28)$$

So

$$\left(x - \frac{1}{2}\right)\omega = \left(\omega^2 - d^2 x\right)^{1/2} \quad -(29)$$

i.e

$$\omega^2\left(x - \frac{1}{2}\right)^2 = \omega^2 - d^2 x \quad -(30)$$

$$\omega^2\left(x^2 - x + \frac{1}{4}\right) = \omega^2 - d^2 x$$

$$x^2\omega^2 + x\left(d^2 - \omega^2\right) + \frac{\omega^2}{4} = 0$$

$$x^2 + \left(\frac{d^2}{\omega^2} - 1\right)x + \frac{1}{4} = 0 \quad -(31)$$

and

$$v^2 = \frac{2\omega}{\alpha^2}\left(1 - x^2\right) \quad -(32)$$

From eq. (32): $$x^2 = 1 + \frac{\alpha^2 v^2}{2\omega} \quad -(33)$$

From eq (31): $$-(34)$$

$$x = \frac{1}{2}\left(\left(1 - \frac{\alpha^2}{v^2\omega^2}\right) \pm \left(\left(1 - \frac{\alpha^2}{v^2\omega^2}\right)^2 - 1\right)^{1/2}\right)$$

So:

6)

$$1 + \frac{\alpha^2 V^2}{2\omega} = \frac{1}{4}\left(1 - \frac{\alpha^2}{V^2\omega^2} \pm \left(\left(1 - \frac{\alpha^2}{V^2\omega^2}\right)^2 - 1\right)^{1/2}\right)^2 \quad -(35)$$

in which

$$\alpha V = d = \left(\frac{N}{V}\right)\frac{|\mu_{fi}|^2}{6\epsilon_0\hbar} \quad -(36)$$

Therefore V can be found in terms of d from eq. (35), which is:

$$\boxed{1 + \frac{d^2}{2\omega} = \frac{1}{4}\left(1 - \frac{d^2}{\omega^2 V^4} \pm \left(\left(1 - \frac{d^2}{\omega^2 V^4}\right)^2 - 1\right)\right)^{1/2}\Bigg)^2} \quad -(37)$$

<u>so V can be found in terms of d and ω.</u>

Finally the photon mass is found from

$$\kappa = \frac{\omega}{V} \quad -(38)$$

in eq. (1), so:

$$\hbar\kappa = \hbar\frac{\omega}{V} = \gamma m v \quad -(39)$$

Therefore:

$$\hbar\omega = \gamma m v^2 \quad -(40)$$

and

$$m = \frac{\hbar\omega}{\gamma v^2} \quad -(41)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(42)$$

so

$$\boxed{m = \frac{\hbar\omega}{v^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \quad -(43)$$

The photon mass can be found from any power absorption coefficient α, observed or calculated.