290(9): Analytical Expression for the Correction to the Stefan Boltzmann Law

From note 290(8), to first order in $d\omega$:

$$\frac{dN}{dN_1} = 1 + \frac{2}{3\omega} d\omega \quad -(1)$$

By definition:

$$\frac{dN}{dN_1} = \lim_{\delta N_1 \to 0} \left(\frac{f(N_1 + \delta N_1) - f(N_1)}{\delta N_1} \right) \quad -(2)$$

where:

$$N = f(N_1) \quad -(3)$$

For example if:

$$N = N_1 \quad -(4)$$

then

$$\frac{dN}{dN_1} = \lim_{\delta N_1 \to 0} \left(\frac{N_1 + \delta N_1 - N_1}{\delta N_1} \right) \quad -(5)$$

$$= \underline{1}$$

Let

$$N = \left(1 + \frac{2}{3\omega} d\omega \right) N_1 \quad -(6)$$

then

$$f(N_1) = \left(1 + \frac{2}{3\omega} d\omega \right) N_1 \quad -(7)$$

and

$$f(N_1 + \delta N_1) = \left(1 + \frac{2}{3\omega} d\omega \right) (N_1 + \delta N_1) \quad -(8)$$

2)

So:
$$\frac{dN}{dN_1} = 1 + \frac{2}{3\omega} d\omega \quad -(9)$$

which is eq. (1).

Therefore the corrected number of photons is given by eq. (6). In the old theory the number of photons is, for a black body:

$$\frac{N_1}{V} = \frac{2\zeta(3)}{\pi^2}\left(\frac{kT}{c\hbar}\right)^3 \quad -(10)$$

where
$$\zeta(3) = 1.20206 \quad -(11)$$

from:
$$\int_0^\infty \left(\frac{x^{2n}}{e^x - 1}\right) dx = (2n)!\,\zeta(2n+1) \quad -(12)$$

Eq. (10) is a form of the Stefan Boltzmann law for number of photons per unit volume rather than intensity. The number of photons is proportional to the cube of temperature from the Planck distribution applied to black body radiation.

In the corrected theory the number of photons per unit volume of black body radiation is:

3)

$$\frac{N}{V} = \left(1 + \frac{2}{3\omega} d\omega\right) \frac{N_1}{V}$$

$$= \frac{2\zeta(3)}{\pi^2}\left(\frac{kT}{c\hbar}\right)^3 \left(1 + \frac{2}{3\omega} d\omega\right) \quad -(13)$$

$$:= \frac{1}{V}(N + dN)$$

So the first order correction to the Stefan Boltzmann law is:

$$\boxed{\frac{1}{V}\frac{dN}{d\omega} = \frac{4\zeta(3)}{3\pi^2}\left(\frac{kT}{c\hbar}\right)^3 \cdot \frac{1}{\omega}} \quad -(14)$$

This correction must be applied to the original Stefan Boltzmann law, which is:

$$\frac{N}{V} = \frac{2\zeta(3)}{\pi^2}\left(\frac{kT}{c\hbar}\right)^3 \quad -(15)$$

so:

$$\boxed{\frac{dN}{d\omega} = \frac{2}{3\omega}} \quad -(16)$$

As $\omega \to 0$ this becomes a large correction.