290(3): Difficulties in the Derivation of the Rayleigh Jeans Density of States.

The derivation originates in the number density of oscillators:

$$\frac{N}{V} = \left(\frac{\omega^3}{6c^3\pi^2}\right) \quad -(1)$$

in the notation of previous notes. The next step is to calculate the infinitesimal of number density:

$$\frac{dN}{V} = \left(\frac{1}{6c^3\pi^2}\right)\left((\omega + d\omega)^3 - \omega^3\right) \quad -(2)$$

The correct expression for Eq. (2) is:

$$\frac{dN}{V} = \frac{1}{6\pi^2c^3}\left(\omega^3 + 2\omega^2 d\omega + \omega(d\omega)^2 + \omega^2 d\omega + (d\omega)^3 - \omega^3\right) \quad -(3)$$

$$= \frac{1}{6\pi^2c^3}\left(3\omega^2 d\omega + 2\omega(d\omega)^2 + (d\omega)^3\right)$$

$$= \frac{\omega^2}{2\pi^2c^3}d\omega + \frac{\omega}{3\pi^2c^3}(d\omega)^2 + \frac{(d\omega)^3}{6\pi^2c^3}$$

In the Rayleigh Jeans theory this result is doubled because it is assumed that there are two senses of polarization, so:

$$\frac{dN}{V} = \frac{\omega^2}{\pi^2c^3}d\omega + \frac{2\omega}{3\pi^2c^3}(d\omega)^2 + \frac{(d\omega)^3}{3\pi^2c^3} \quad -(4)$$

2) In the usual theory the last two terms are omitted,

so:

$$\frac{dN}{V} = \frac{\omega^2}{\pi^2 c^3} \cdot d\omega \quad -(5)$$

and this is the usual Rayleigh Jeans density of states.

The usual infinitesimal of energy density is:

$$\frac{dU}{V} = \langle E \rangle \frac{dN}{V} \quad -(6)$$

where $\langle E \rangle$ is the average energy of the Planck oscillator:

$$\langle E \rangle = \left(\frac{x}{1-x} \right) \hbar \omega \quad -(7)$$

where:

$$x = \exp\left(-\frac{\hbar \omega}{kT} \right) \quad -(8)$$

So

$$\frac{dU_0}{V} = \frac{\hbar \omega^3}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega \quad -(9)$$

which is the usual Planck distribution.

The Stefan Boltzmann law is found from:

$$\frac{U_0}{V} = \int_0^\infty \frac{\hbar \omega^3}{\pi^2 c^3} \left(\frac{x}{1-x} \right) d\omega \quad -(10)$$

$$= \left(\frac{\pi^2 k^4}{15 c^3 \hbar^3} \right) T^4$$

The new density is:

$$\therefore \quad I = \frac{c\,U_0}{V} = \left(\frac{\pi^2 k^4}{15 c^2 \hbar^3}\right) T^4 \quad -(11)$$

However the correct derivation is:

$$\frac{dU}{V} = \langle E \rangle \left(\frac{\omega^2}{\pi^2 c^3} d\omega + \frac{2\omega}{3\pi^2 c^3} (d\omega)^2 + \frac{(d\omega)^3}{3\pi^2 c^3} \right)$$

$$-(12)$$

So:

$$\frac{U}{V} = \int \frac{\langle E \rangle \omega^2}{\pi^2 c^3} d\omega + \frac{2}{3\pi^2 c^3} \int \langle E \rangle \omega (d\omega)^2 + \frac{1}{3\pi^2 c^3} \int \langle E \rangle (d\omega)^3$$

$$-(13)$$

The second two integrals are evaluated as follows. Consider firstly:

$$\frac{dU_1}{V} = \frac{2\omega \langle E \rangle}{3\pi^2 c^3} (d\omega)^2 \quad -(14)$$

then

$$\left(\frac{dU_1}{V}\right)^{1/2} = \left(\frac{2}{3\pi^2 c^3}\right)^{1/2} \left(\omega^{1/2} \langle E \rangle^{1/2} d\omega\right)$$

$$-(15)$$

so

$$\frac{dU_1}{V} = \frac{2}{3\pi^2 c^3} \left(\omega^{1/2} \langle E \rangle^{1/2} d\omega\right)^2 \quad -(16)$$

Therefore:

4)

$$\frac{U_1}{V} = \frac{2}{3\pi^2 c^3} \int \omega \langle E \rangle (d\omega)^2 \quad - (17)$$

$$= \frac{2}{3\pi^2 c^3} \iint \omega \langle E \rangle \, d\omega \, d\omega$$

$$= \frac{2}{3\pi^2 c^3} \int \left(\int \omega \langle E \rangle \, d\omega \right) d\omega$$

Secondly:

$$\frac{U_2}{V} = \frac{1}{3\pi^2 c^3} \int \langle E \rangle (d\omega)^3 \quad - (18)$$

$$= \frac{1}{3\pi^2 c^3} \int \langle E \rangle \, d\omega \, d\omega \, d\omega$$

$$= \frac{1}{3\pi^2 c^3} \int \left(\int \left(\int \langle E \rangle \, d\omega \right) d\omega \right) d\omega$$

The total energy density is:

$$\frac{U}{V} = \frac{U_0}{V} + \frac{U_1}{V} + \frac{U_2}{V} \quad - (19)$$
