280(4): Total Internal Reflection (Optical Fibres)

Consider Snell's law:

$$n \sin\theta = n_1 \sin\theta_1 \quad - (1)$$

where θ is the angle of incidence and θ_1 is the angle of refraction. Here n is the refractive index of the medium of the incident beam and n_1 that of the refracted beam. Total internal reflection occurs at:

$$\theta_1 = \frac{\pi}{2} \quad - (2)$$

under the condition:

$$n_1 < n. \quad - (3)$$

In general:

$$\underline{\kappa} = \kappa(\underline{i}\sin\theta + \underline{j}\cos\theta) - (4)$$

$$\underline{\kappa_1} = \kappa_1(\underline{i}\sin\theta_1 + \underline{j}\cos\theta_1) - (5)$$

$$\underline{\kappa_2} = \kappa_2(\underline{i}\sin\theta_2 - \underline{j}\cos\theta_2) - (6)$$

in the notation of the previous note. For total internal reflection:

$$\underline{\kappa_1} = \kappa_1 \underline{i} \quad - (7)$$

and

$$\underline{\kappa_2} = \kappa_2(\underline{i}\sin\theta - \underline{j}\cos\theta) - (8)$$

where we have used:

$$\theta = \theta_2. - (9)$$

2) The medium of reflection is the same as the medium of incidence, so

$$\kappa = \frac{\omega}{v} \quad -(10)$$

$$\kappa_2 = \frac{\omega_2}{v} \quad -(11)$$

and

$$\kappa_1 = \frac{\omega_1}{c} \quad -(12)$$

For total internal reflection to occur eq. (3) must be satisfied so the refractive index of the incoming medium must be higher than that of the medium of refraction. The medium of refraction in eq. (12) has been chosen to be air. Eq. (7) shows that refraction takes place along the $\underline{i}$ axis

From eq. (1):

$$\sin\theta = \frac{n_1}{n} \quad -(13)$$

and if the medium of refraction is air ($n_1 = 1$):

$$\sin\theta = \frac{1}{n} \quad -(14)$$

so

$$\cos\theta = \left(1 - \frac{1}{n^2}\right)^{1/2} \quad -(15)$$

Therefore:

$$\underline{\kappa} = \kappa\left(\frac{\underline{i}}{n} + \left(1 - \frac{1}{n^2}\right)^{1/2}\underline{j}\right)$$

$$= \frac{\kappa}{n}\left(\underline{i} + \left(n^2 - 1\right)^{1/2}\underline{j}\right)$$

$-(16)$

3)

$$\underline{\kappa}_1 = \kappa_1 \underline{i} \quad -(17)$$

and

$$\underline{\kappa}_2 = \frac{\kappa_2}{n}\left(\underline{i} - (n^2-1)^{1/2}\underline{j}\right) \quad -(18)$$

By conservation of momentum:

$$\underline{\kappa} = \underline{\kappa}_1 + \underline{\kappa}_2 \quad -(19)$$

in the one photon theory and

$$\langle \underline{\kappa} \rangle = \langle \underline{\kappa}_1 \rangle + \langle \underline{\kappa}_2 \rangle \quad -(20)$$

in the n photon theory.

By conservation of energy:

$$\omega = \omega_1 + \omega_2 \quad -(21)$$

in the one photon theory and

$$\langle \omega \rangle = \langle \omega_1 \rangle + \langle \omega_2 \rangle \quad -(22)$$

in the n photon theory.

From eq. (19):

$$\underline{\kappa}_1 = \underline{\kappa} - \underline{\kappa}_2 \quad -(23)$$

so:

$$\underline{\kappa}_1 \cdot \underline{\kappa}_1 = (\underline{\kappa} - \underline{\kappa}_2)\cdot(\underline{\kappa} - \underline{\kappa}_2) = \kappa^2 + \kappa_2^2 - 2\kappa\kappa_2 \cos\theta_3 \quad -(24)$$

where

$$\theta_3 = 2\theta \quad -(25)$$

4) Therefore:

$$K_1^2 = K^2 + K_2^2 - 2KK_2 \cos 2\theta \quad -(26)$$

i.e

$$\left(\frac{\omega_1}{c}\right)^2 = \left(\frac{\omega}{v}\right)^2 + \left(\frac{\omega_2}{v}\right)^2 - \frac{2\omega\omega_2}{v^2}\cos 2\theta \quad -(27)$$

where

$$n = \frac{c}{v} \quad -(28)$$

is the refractive index of the medium of incidence, for example the glass of an optical fibre.

So

$$\omega_1^2 = n^2\left(\omega^2 + \omega_2^2 - 2\omega\omega_2 \cos 2\theta\right) \quad -(29)$$

$$= (\omega - \omega_2)^2$$

Here:

$$\cos 2\theta = 2\cos^2\theta - 1 \quad -(30)$$

$$= 1 - \sin^2\theta$$

$$= 1 - \frac{1}{n^2}$$

where we have used eq. (14).

Therefore:

$$(\omega - \omega_2)^2 = n^2\left(\omega^2 + \omega_2^2 - 2\omega\omega_2\left(1 - \frac{1}{n^2}\right)\right)$$

$$-(31)$$

$$= n^2\left((\omega-\omega_2)^2 + 2\omega\omega_2\right) \quad -(32)$$

Finally:

$$(\omega-\omega_2)^2(1-n^2) = 2\omega\omega_2 \quad -(33)$$

i.e.

$$\omega^2 + \omega_2^2 - 2\omega\omega_2 = \left(\frac{2}{1-n^2}\right)\omega\omega_2 \quad -(34)$$

or

$$\omega_2^2 - 2\left(1 + \frac{1}{1-n^2}\right)\omega\omega_2 + \omega^2 = 0, \quad -(35)$$

$$\omega_2^2 - 2\left(\frac{2-n^2}{1-n^2}\right)\omega\omega_2 + \omega^2 = 0, \quad -(36)$$

$$-(37)$$

So

$$\omega_2 = \frac{1}{2}\left(2\left(\frac{2-n^2}{1-n^2}\right)\omega \pm \left(4\left(\frac{2-n^2}{1-n^2}\right)^2 - 4\right)^{1/2}\omega\right)$$

i.e.

$$\omega_2 = \omega\left[\left(\frac{2-n^2}{1-n^2}\right) \pm \left(\left(\frac{2-n^2}{1-n^2}\right)^2 - 1\right)^{1/2}\right] \quad -(38)$$

$$= \frac{\omega}{1-n^2}\left[2-n^2 \pm \left((2-n^2)^2 - (1-n^2)^2\right)^{1/2}\right]$$

i.e

$$\omega_2 = \frac{\omega}{1-n^2}\left[2-n^2 \pm \left((2-n^2)^2 - (1-n^2)^2\right)^{1/2}\right]$$

$$-(39)$$

6) Finally the refracted frequency is found from:

$$\omega_1 = \omega - \omega_2 \qquad -(40)$$

For the glass of an optical fibre:

$$n = 1.5 \qquad -(41)$$

So

$$\omega_2 = 0.2\omega \mp \frac{1.5^{1/2}}{1.5} i \qquad -(42)$$

and

$$\boxed{\begin{aligned} \text{Real } \omega_2 &= 0.2\omega \\ \omega_1 &= 0.8\omega \end{aligned}} \qquad -(43)$$