280(1): Theory of Microwave Reflection from Water.

Conservation of energy and momentum for each photon gives:

$$\omega = \omega_1 + \omega_2 \quad -(1)$$

$$\underline{\kappa} = \underline{\kappa_1} + \underline{\kappa_2} \quad -(2)$$

where ω is the incident angular frequency, ω_1 is the refracted angular frequency and ω_2 the reflected angular frequency. Here $\underline{\kappa}$ is the incident wave vector, $\underline{\kappa_1}$ the refracted wave vector and $\underline{\kappa_2}$ the reflected wave vector.

In an n photon monochromatic beam:

$$\left(\frac{x}{1-x}\right)\omega = \left(\frac{x_1}{1-x_1}\right)\omega_1 + \left(\frac{x_2}{1-x_2}\right)\omega_2 \quad -(3)$$

where

$$x = \exp\left(-\frac{\hbar\omega}{kT}\right) \quad -(4)$$

and so on.

Now use:

$$\langle \hbar\underline{\kappa_1}\rangle = \langle \hbar\underline{\kappa}\rangle - \langle \hbar\underline{\kappa_2}\rangle \quad -(5)$$

so:

$$\langle\underline{\kappa_1}\rangle\cdot\langle\underline{\kappa_1}\rangle = \langle\underline{\kappa}\rangle\cdot\langle\underline{\kappa}\rangle + \langle\underline{\kappa_2}\rangle\cdot\langle\underline{\kappa_2}\rangle - 2\langle\underline{\kappa}\rangle\cdot\langle\underline{\kappa_2}\rangle\cos 2\theta \quad -(6)$$

because the angle between $\underline{\kappa}$ and $\underline{\kappa_2}$ is twice the incident angle θ by Snell's Law.

2) So:

$$\left(\frac{x_1}{1-x_1}\right)^2\left(\frac{\omega_1}{V_1}\right)^2 = \left(\frac{x}{1-x}\right)^2\left(\frac{\omega}{c}\right)^2 + \left(\frac{x_2}{1-x_2}\right)^2\left(\frac{\omega_2}{c}\right)^2$$
$$- 2\left(\frac{x}{1-x}\right)\left(\frac{x_2}{1-x_2}\right)\frac{\omega\omega_2}{c^2}\cos(2\theta) \quad -(7)$$

In the linearized theory:

$$A_1\omega_1 = A\omega - A_2\omega_2 \quad -(8)$$

and

$$A_1^2\left(\frac{\omega_1}{V_1}\right)^2 = A^2\left(\frac{\omega}{c}\right)^2 + A_2^2\left(\frac{\omega_2}{c}\right)^2 - 2AA_2\omega\omega_2\cos(2\theta)/c^2 \quad -(9)$$

where

$$\frac{1}{V_1^2} = \frac{n_1^2}{c^2} \quad -(10)$$

where n_1 is the refractive index of the medium from which reflection takes place, in this case water.

So:

$$A_1^2\omega_1^2 n_1^2 = A^2\omega^2 + A_2^2\omega_2^2$$
$$- 2AA_2\omega\omega_2\cos(2\theta) \quad -(11)$$

Here:

$$A = 1 - \frac{\hbar\omega}{kT} \quad -(12)$$

and so on..

Eliminate ω_1 between eqs. (8) and (11) to find ω_2 in terms of ω and θ.

3) It is known experimentally that the frequency of microwave radiation reflected from water is different from the incident frequency. In the microwave only frequency can be measured, not wavelength. Obviously the diagram:

$$\omega = ?\ \omega_2 \quad -(13)$$

is incorrect.

In order to find the refractive index we

$$n_1'^2 = \frac{1}{2}\left(\epsilon_{1r}' + \left(\epsilon_{1r}'^2 + \epsilon_{1r}''^2\right)^{1/2}\right) \quad -(14)$$

and the Debye theory of dielectric relaxation:

$$\epsilon_{1r}' = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1+\omega_1^2\tau^2} \quad -(15)$$

$$\epsilon_{1r}'' = \frac{(\epsilon_0 - \epsilon_\infty)\,\omega_1\tau}{1+\omega_1^2\tau^2} \quad -(16)$$

where τ is the Debye relaxation time.

In eqs. (15) and (16) the frequency ω_1 in the refracting medium, water, has to be used:

$$A_1\omega_1 = A\omega - A_2\omega_2, \quad -(17)$$

From eqs. (11) and (14):

4)
$$\frac{A_1^2 \omega_1^2}{2}\left(\epsilon_{1r}' + \left(\epsilon_{1r}'^2 + \epsilon_{1r}''^2\right)^{1/2}\right)$$
$$= A^2\omega^2 + A_2^2\omega_2^2 - 2AA_2\omega\omega_2\cos(2\theta) \quad -(18)$$

where ϵ_{1r}' and ϵ_{1r}'' are given by eqs. (15) and (16). Finally ω_1 has to be eliminated between eqs. (8) and (18).

It may be possible for Maxima to solve eqs. (8) and (18) without further approximation. However, if

$$\epsilon_0 \gg \epsilon_\infty \quad -(19)$$

then:

$$n_1^2 \sim \frac{\epsilon_0}{1+\omega_1^2\tau^2} \quad -(20)$$

and:

$$\frac{A_1^2\omega_1^2\epsilon_0}{1+\omega_1^2\tau^2} = A^2\omega^2 + A_2^2\omega_2^2 - 2AA_2\omega\omega_2\cos(2\theta) \quad -(21)$$

$$A_1\omega_1 = A\omega - A_2\omega_2 \quad -(22)$$

For water, eq. (19) is a very good approximation.

Therefore from eqs. (21) and (22):

$$\epsilon_0 \left(A\omega - A_2\omega_2 \right) = \left(1 + \omega_1^2 \tau^2\right)\left(A^2\omega^2 + A_2^2\omega_2^2 - 2AA_2\omega\omega_2 \cos(2\theta)\right) \quad -(23)$$

where:

$$\omega_1^2 = \frac{1}{A_1^2}\left(A\omega - A_2\omega_2\right) \quad -(24)$$

and

$$A_1 = 1 - \frac{\hbar\omega_1}{kT} \quad -(25)$$

Maxima may be able to solve eq. (23) for ω_2 in terms of ω. If not, eq. (25) may be simplified in the first approximation to:

$$A_1 \sim 1 \quad -(26)$$

So:

$$\epsilon_0\left(A\omega - A_2\omega^2\right) = \left(1 + \left(A\omega - A_2\omega_2\right)^2\tau^2\right)\left(A^2\omega^2 + A_2^2\omega_2^2 - 2AA_2\omega\omega_2\cos(2\theta)\right) \quad -(27)$$

This can be solved for ω_2 in terms of ω, θ, ϵ_0 and τ. For water: $\epsilon_0 = 80.1$ at $293K$