280(3): Brewster Angle Refraction and Conservation of Energy and Momentum.

Consider reflection and refraction at an interface and let the incident angle be θ, the angle of refraction be θ_1 and the angle of reflection by θ_2. By Snell's law:

$$\theta = \theta_2 \quad - (1)$$

and

$$n \sin\theta = n_1 \sin\theta_1 \quad - (2)$$

where n is the refractive index of the incident and reflected medium, and n_1 is the refractive index of the medium of refraction.

The Brewster angle is defined by:

$$\theta_B = \tan^{-1}\left(\frac{n_1}{n}\right) \quad - (3)$$

From eqs. (2) and (3):

$$\theta + \theta_1 = \frac{\pi}{2} \quad - (4)$$

at the Brewster angle of incidence:

$$\theta = \theta_B. \quad - (5)$$

This is because:

$$n \sin\theta_B = n_1 \sin\left(\frac{\pi}{2} - \theta_B\right) \quad - (6)$$

$$= n_1 \cos\theta_B$$

and so

$$\frac{\sin\theta_B}{\cos\theta_B} = \tan\theta_B = \frac{n_1}{n} \quad - (7)$$

which is eq. (3) QED. For air, $n \sim 1$, for glass, $n_1 \sim 1.5$, so $\theta_B = 56^\circ$. At the Brewster angle there is no reflection.

2)

In general, if $\underline{K}$ is the incident wave vector, $\underline{K_1}$ the refracted wave vector, and $\underline{K_2}$ the law of conservation of momentum for one photon demands that:

$$\hbar \underline{K} = \hbar \underline{K_1} + \hbar \underline{K_2} \quad -(8)$$

where:

$$\underline{K} = K\left(\underline{i}\sin\theta + \underline{j}\cos\theta\right) \quad -(9)$$

$$\underline{K_1} = K_1\left(\underline{i}\sin\theta_1 + \underline{j}\cos\theta_1\right) \quad -(10)$$

$$\underline{K_2} = K_2\left(\underline{i}\sin\theta_2 - \underline{j}\cos\theta_2\right) \quad -(11)$$

At the Brewster angle:

$$\theta = \theta_2 \quad -(12)$$

and

$$\theta + \theta_1 = \frac{\pi}{2} \quad -(13)$$

Therefore at the Brewster angle:

$$\underline{K} = \frac{\omega}{c}\left(\underline{i}\sin\theta_B + \underline{j}\cos\theta_B\right) - (14)$$

$$\underline{K_1} = \frac{\omega_1}{v}\left(\underline{i}\cos\theta_B + \underline{j}\sin\theta_B\right) - (15)$$

$$K_2 = \frac{\omega_2}{c}\left(\underline{i}\sin\theta_B - \underline{j}\cos\theta_B\right) - (16)$$

where we have used:

$$K = \frac{\omega}{c}, \quad K_1 = \frac{\omega_1}{v}, \quad K_2 = \frac{\omega_2}{c} \quad -(17)$$

Now use:

$$\underline{K} = \underline{K_1} + \underline{K_2} \quad -(18)$$

) For the $\underline{i}$ components:

$$\frac{\omega}{c}\sin\theta_B = \frac{\omega_1}{v}\cos\theta_B + \frac{\omega_2}{c}\sin\theta_B \quad -(19)$$

and for the $\underline{j}$ components:

$$\frac{\omega}{c}\cos\theta_B = \frac{\omega_1}{v}\sin\theta_B - \frac{\omega_2}{c}\cos\theta_B \quad -(20)$$

From eqs. (19) and (20):

$$\frac{\omega^2}{c^2} = \left(\frac{\omega_1}{v}\cos\theta_B + \frac{\omega_2}{c}\sin\theta_B\right)^2 + \left(\frac{\omega_1}{v}\sin\theta_B - \frac{\omega_2}{c}\cos\theta_B\right)^2 \quad -(21)$$

$$= \left(\frac{\omega_1}{v}\right)^2\cos^2\theta_B + \left(\frac{\omega_2}{c}\right)^2\sin^2\theta_B + \frac{2\omega_1\omega_2}{cv}\cos\theta_B\sin\theta_B$$

$$+ \left(\frac{\omega_1}{v}\right)^2\sin^2\theta_B + \left(\frac{\omega_2}{c}\right)^2\cos^2\theta_B - \frac{2\omega_1\omega_2}{cv}\cos\theta_B\sin\theta_B$$

$$= \frac{\omega_1^2}{v^2} + \frac{\omega_2^2}{c^2}$$

If there is no reflection then:

$$\omega_2 = 0 \quad -(22)$$

so

$$\boxed{\frac{\omega}{c} = \frac{\omega_1}{v}} \quad -(23)$$

Eq. (23) can only be true if

$$v = ? c \quad -(24)$$

4) which is inconsistent with the fact that:

$$v \neq c \quad -(25)$$

Therefore the n photon monochromatic theory must be used and:

$$\frac{\langle \omega \rangle}{c} = \frac{\langle \omega_1 \rangle}{v} \quad -(26)$$

i.e.

$$\boxed{\frac{\omega}{c}\left(\frac{x}{1-x}\right) = \frac{\omega_1}{v}\left(\frac{x_1}{1-x_1}\right)} \quad -(27)$$

where:

$$x = \exp\left(-\frac{\hbar\omega}{kT}\right) \quad -(28)$$

$$x_1 = \exp\left(-\frac{\hbar\omega_1}{kT}\right) \quad -(29)$$

and

$$n_1 = \frac{c}{v} \quad -(30)$$

So at the Brewster angle ω_1 can be found in terms of ω using eq. (27) to eq. (30) and:

$$\langle \hbar\omega_2 \rangle = 0 \quad -(31)$$

i.e.

$$\left(\frac{x_2}{1-x_2}\right)\hbar\omega_2 = 0 \quad -(32)$$

where:

$$x_2 = \exp\left(-\frac{\hbar\omega_2}{kT}\right) \quad -(33)$$

5) Eq. (37) is conservation of momentum:

$$\langle \hbar \underline{\kappa} \rangle = \langle \hbar \underline{\kappa}_1 \rangle \quad -(34)$$

and this must be considered with conservation of energy:

$$\langle \hbar \omega \rangle = \langle \hbar \omega_1 \rangle \quad -(35)$$

with

$$v \neq c. \quad -(36)$$

The only way this is possible is to use the de Broglie / Einstein equations:

$$\hbar \omega = \gamma m c^2 \quad -(37)$$

and

$$\hbar \underline{\kappa} = \gamma m \underline{v} \quad -(38)$$

where m is the mass of the photon and:

$$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \quad -(39)$$

Therefore for one photon:

$$\hbar \omega = \hbar \omega_1 = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} m c^2 \quad -(40)$$

and

$$\omega_2 = 0 \quad -(41)$$

at the Brewster angle.

6) The photon mass is therefore:

$$m = \frac{\hbar\omega}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad -(42)$$

where the refractive index is:

$$n = \frac{c}{v} \quad -(43)$$

Also:

$$\hbar \underline{\kappa} = \hbar \underline{\kappa_1} = \gamma m \underline{v} \quad -(44)$$

and

$$\hbar \underline{\kappa_2} = \underline{0} \quad -(45)$$

at the Brewster angle.