211(4): A New Formulation of the Schroedinger Equation from 3D Conic Sections

Consider the classical hamiltonian of 3D orbit theory:

$$H = \frac{p^2}{2m} + U(r) = E \quad -(1)$$

where

$$\frac{p^2}{2m} = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) \quad -(2)$$

and

$$U = -\frac{k}{r} \quad -(3)$$

as in previous notation.

In gravitational theory:

$$k = mMG \quad -(4)$$

and in atomic theory:

$$k = \frac{e^2}{4\pi \epsilon_0} \quad -(5)$$

The quantization condition is:

$$-i\hbar \underline{\nabla} \phi = \underline{p} \phi \quad -(6)$$

so the Schroedinger equation is:

$$-\frac{\hbar^2}{2m} \nabla^2 \phi = (E - U)\phi = T\phi$$
$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) \phi \quad -(7)$$

where T is the kinetic energy.

The classical hamiltonian (1) is rigorously the same as the beta conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(8)$$

where:

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(9)$$

and where the conserved total angular momentum is

$$L = mr^2\dot{\beta} \quad -(10)$$

Using:

$$\frac{dr}{dt} = \frac{dr}{d\beta}\frac{d\beta}{dt} \quad -(11)$$

the kinetic energy can be expressed as:

$$T = \frac{L^2}{2mr^4}\left(\left(\frac{dr}{d\beta}\right)^2 + r^2\right) \quad -(12)$$

where

$$\frac{dr}{d\beta} = -\frac{\epsilon r^2}{\alpha}\sin\beta \quad -(13)$$

It follows that:

$$T = \frac{1}{2}\frac{L^2}{m\alpha^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right) \quad -(14)$$

giving a new type of Schrodinger equation:

$$\boxed{\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\phi &= (E-U)\phi \\ &= \frac{L^2}{2m\alpha^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right)\phi \end{aligned}} \quad -(15)$$

and a new characterization of atoms and molecules.

In atoms and molecules:

$$\alpha = \frac{L^2}{mk} = \frac{4\pi\epsilon_0 L^2}{me^2} \quad -(16)$$

The particle on a sphere is given by

$$U = 0, \quad \epsilon = 0, \quad \alpha = r \quad -(17)$$

in eq (15), which reduces to:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = E\phi = \frac{L^2}{2m\alpha^2}\phi \quad -(18)$$

i.e.

$$-\frac{\hbar^2\Lambda^2}{r^2}\phi = \frac{L^2}{\alpha^2}\phi \quad -(19)$$

4) Using:

$$r = \alpha \quad -(20)$$

and

$$\Lambda^2 \phi = -\ell(\ell+1)\phi \quad -(21)$$

it is found that:

$$L^2 = \ell(\ell+1)\hbar^2 \quad -(22)$$

for the particle on a sphere.

For the Bohr atom:

$$L = n\hbar \quad -(23)$$

and

$$\epsilon = 0, \quad U = -k/r \quad -(24)$$

with

$$\alpha = r \quad -(25)$$

So the Bohr atom is characterized by:

$$n^2 = \ell(\ell+1) \quad -(26)$$

In general eq. (15) is:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = (E-U)\phi$$

$$= \frac{nk^2}{2L^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right) \quad -(27)$$

5) i.e.

$$-\frac{\hbar^2}{2m}\nabla^2\phi = (E-U)\phi \qquad -(28)$$

$$= \frac{me^4}{32\pi^2\epsilon_0^2 L^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right)\phi$$

Therefore the energy levels of the H atom are given by:

$$E\phi = \left[\frac{me^4}{32\pi^2\epsilon_0^2 L^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right) - \frac{k}{r}\right]\phi$$

$$= \hat{H}\phi \qquad -(29)$$

It follows that: $\qquad -(30)$

$$\left\langle \frac{me^4}{32\pi^2\epsilon_0^2 L^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right)\right\rangle = \frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 n^2}$$

where n is the principal quantum number.

so $$\left\langle \frac{1+\epsilon^2+2\epsilon\cos\beta}{L^2}\right\rangle = \frac{1}{n^2\hbar^2} \qquad -(31)$$

so :

6)
$$\langle L^2 \rangle = \frac{n^2 \hbar^2}{1+\epsilon^2+2\epsilon\cos\beta} \quad -(32)$$

when
$$\epsilon = 0 \quad -(33)$$

this reduces to:
$$\langle L^2 \rangle = n^2 \hbar^2 \quad -(34)$$

which is the Bohr quantization.

In general there are expectation values of $\cos\beta$ defined by:
$$\langle \cos\beta \rangle = \int \phi^* \cos\beta \, \phi \, d\tau \quad -(35)$$

where:
$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L}{L_Z}\right)^2 \sin^2\phi\right)^{1/2}}$$
$$= \left(1 - \left(\frac{1}{1-\left(\frac{L_Z}{L}\right)^2}\right)\cos^2\theta\right)^{1/2} \quad -(36)$$
