277(3): The Bohr Atom and Three Dimensional Orbit Theory: The Quantized Spherical Orbit.

The Bohr atom is a three dimensional theory based on the classical hamiltonian:

$$H = E = \frac{1}{2} m v^2 - \frac{k}{r} \quad -(1)$$

where:

$$k = \frac{e^2}{4\pi \epsilon_0} \quad -(2)$$

and

$$v^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \quad -(3)$$

The hamiltonian (1) is equivalent to the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} \quad -(4)$$

in general. The force equation corresponding to eq. (1) is found from the lagrangian:

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{k}{r} \quad -(5)$$

and the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(6)$$

which gives:

$$m \ddot{r} = \frac{L^2}{m r^3} - \frac{k}{r^2} \quad -(7)$$

which is the Leibniz equation for gravitation with d)

$$k = mMG \quad -(8)$$

replaced by eq. (2).

The Bohr theory of the atom uses the turning point:

$$m\ddot{r} = 0 \quad -(9)$$

at which:

$$\frac{L^2}{mr^3} = \frac{k}{r^2}. \quad -(10)$$

The hamiltonian equivalent to the turning point is:

$$H = E = r^2 \dot{\beta}^2 - \frac{k}{r} \quad -(11)$$

and the turning point is defined by:

$$\frac{dr}{dt} = 0 \quad -(12)$$

The hamiltonian (11) is that of the particle on a sphere with the addition of the potential:

$$U = -\frac{k}{r}. \quad -(13)$$

Using:

$$\frac{dr}{dt} = \frac{dr}{d\beta}\frac{d\beta}{dt} \quad -(14)$$

the conic section for the Bohr theory can be found. Considering the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) \quad -(15)$$

with the Lagrangian (5) it is found that:

$$\frac{d}{dt}\left(mr^2\dot{\beta}\right) = 0 \quad -(16)$$

so the conserved total angular momentum L is

$$L = mr^2\dot{\beta} \quad -(17)$$

therefore

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(18)$$

$$\neq 0$$

It follows from eqns. (12) and (14) that:

$$\frac{dr}{d\beta} = 0 \quad -(19)$$

at the turning point (9). Eq. (19) is true if

$$\epsilon = 0 \quad -(20)$$

in the beta conic section (4). Therefore the conic section of the Bohr theory is the sphere:

$$r = \alpha. \quad -(21)$$

From eqns (2) and (10):

4)

$$r = \frac{4\pi\epsilon_0 L^2}{me^2} \quad -(22)$$

The Bohr quantization is :

$$L = n\hbar \quad -(23)$$

where n is an integer. The Bohr radius is defined by:

$$r_B = \frac{4\pi\epsilon_0 \hbar^2}{me^2} \quad -(24)$$

so

$$r = n^2 r_B \quad -(25)$$

are the radii of the Bohr atom.

It follows that :

$$\dot{\beta} = \frac{L}{mr^2} = \frac{\hbar}{mn^3 r_B^2} \quad -(26)$$

and

$$\boxed{\beta = \left(\frac{\hbar}{mn^3 r_B^2}\right) t} \quad -(27)$$

The corresponding result for the particle on a sphere is :

$$\boxed{\beta = \frac{\hbar}{I} \left(\ell(\ell+1)\right)^{1/2} t} \quad -(28)$$

as in note 277(1).

5) The Hamiltonian of the Bohr atom is:

$$E = H = \frac{1}{2} m r^2 \dot{\phi}^2 - \frac{k}{r} \quad -(29)$$

$$= \frac{p^2}{2m} - \frac{k}{r}$$

so the Schroedinger quantization corresponding to the Bohr atom is:

$$-\frac{\nabla^2 \hbar^2}{2m} \phi = \frac{1}{2} m r^2 \dot{\phi}^2 \quad -(30)$$

Eq. (30) is the same as that of the particle on a sphere, but in the Bohr atom:

$$\hat{H} \phi = E \phi \quad -(31)$$

where

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 - \frac{k}{r} \quad -(32)$$

In the particle on a sphere there is no potential energy term.

Therefore the Bohr atom is a quantized spherical orbit.