274(8): Ellipsoidal Orbit

From the last note:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(1)$$

$$\frac{x^2}{a^2} + \frac{z^2}{b^2\left(1-\left(\frac{L_Z}{L}\right)^2\right)} = 1 \quad -(2)$$

$$z^2 = \left(1 - \frac{L_Z}{L}\right) y^2 \quad -(3)$$

1) Adding eqns (1) and (2):

$$\frac{x^2}{a^2} + \frac{y^2}{2b^2} + \frac{z^2}{2b^2\left(1-\left(\frac{L_Z}{L}\right)^2\right)} = 1 \quad -(4)$$

This is the ellipsoid:

$$\boxed{\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1} \quad -(5)$$

where:

$$A = a \quad -(6)$$

2)

$$B = \sqrt{2}\, b \quad -(7)$$

$$C = \sqrt{2}\, b \left(1 - \frac{L_Z}{L}\right) \quad -(8)$$

Therefore the inverse square law of attraction gives an ellipsoidal orbit in general.

Here a and b are the semi major and semi minor axes of the beta ellipse, L is the total angular momentum, and L_Z the magnitude of the angular momentum in the Z axis.

2) Subtract Eq. (3) from Eq. (1):

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - \frac{Z^2}{c^2} = 1 - \left(\frac{L_Z}{L}\right)\frac{Y^2}{c^2} \quad -(9)$$

i.e.:

$$\frac{X^2}{a^2} + Y^2\left(\frac{1}{b^2} + \left(\frac{L_Z}{L}\right)\frac{1}{c^2}\right) - \frac{Z^2}{c^2} = 1$$

$$-(10)$$

This is the one sheet hyperboloid:

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} - \frac{Z^2}{C^2} = 1 \quad -(11)$$

3) where:

$$A = a \quad -(12)$$

$$\frac{1}{B^2} = \frac{1}{b^2} + \left(\frac{L_z}{L}\right)\frac{1}{c^2} \quad -(13)$$

$$C = c \quad -(14)$$

3) Subtract the root of eq. (3) from eq. (1) to obtain the elliptic paraboloid:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = \frac{Z}{c} \quad -(15)$$

In these equations:

$$X = a\epsilon + r\cos\beta \quad -(16)$$

$$Y = r\sin\beta \quad -(17)$$

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(18)$$

$$\alpha = a(1-\epsilon^2) \quad -(19)$$

$$\epsilon^2 = 1 - \frac{b^2}{a^2} \quad -(20)$$

$$H = \frac{1}{2}m(\dot{r}^2 + \dot{\beta}^2 r^2) - \frac{k}{r} \quad -(21)$$

$$\dot{\beta} = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(22)$$