274(5): Recalculation of Note 274(1)

The equation to be solved is:

$$\frac{\cos^2\phi}{\cos^2\phi + y\sin^2\phi} + \frac{\cos^2\theta}{1-y} = 1 \quad -(1)$$

i.e.

$$(1-y)\cos^2\phi + \cos^2\theta\left(\cos^2\phi + y\sin^2\phi\right)$$
$$= (1-y)\left(\cos^2\phi + y\sin^2\phi\right) \quad -(2)$$

or

$$\cos^2\theta\left(\cos^2\phi + y\sin^2\phi\right) = y(1-y)\sin^2\phi$$

$$y^2\sin^2\phi = y\left(\sin^2\phi - \cos^2\theta\sin^2\phi\right) - \cos^2\theta\cos^2\phi$$

$$y^2 + \left(\cos^2\theta - 1\right)y + \frac{\cos^2\phi\cos^2\theta}{\sin^2\phi} = 0. \quad -(3)$$

Therefore:

$$y = \frac{1}{2}\left[1 - \cos^2\theta + \left(\left(1-\cos^2\theta\right)^2 - 4\frac{\cos^2\theta}{\tan^2\phi}\right)^{1/2}\right]$$

$$= \frac{L_Z^2}{L^2} \quad -(4)$$

This gives the correct 2D limit:

$$y \rightarrow 1 \text{ as } \theta \rightarrow \frac{\pi}{2} \quad -(5)$$

2) In 3D the total angular momentum is conserved:

$$L^2 = m^2 r^4 \left(\dot{\phi}^2 \sin^4\theta + \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta \right) \quad -(6)$$

= constant

but L_z is not conserved:

$$L_z^2 = m^2 r^4 \sin^4\theta \, \dot{\phi}^2 \quad -(7)$$

From these equations:

$$\left(\frac{L_z}{L}\right)^2 = \frac{\sin^4\theta \, \dot{\phi}^2}{\sin^4\theta + \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta}$$

$$= \frac{m r^2 \sin^4\theta \, \dot{\phi}^2}{L^2} \quad -(8)$$

Therefore $(L_z / L)^2$ is proportional to $\sin^4\theta$ and $\dot{\phi}^2$. From eqns. (4) and (8):

$$\frac{\sin^4\theta \, \dot{\phi}^2}{\sin^4\theta + \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta} \quad -(9)$$

$$= \frac{1}{2}\left[(1 - \cos^2\theta) + \left((1 - \cos^2\theta)^2 - \frac{4\cos^2\theta}{\tan^2\phi} \right)^{1/2} \right]$$