274(1): Derivation of the Beta Binet Equation

In spherical polar coordinates the hamiltonian is:

$$H = \frac{1}{2} m v^2 + U \quad - (1)$$

and the lagrangian is:

$$\mathcal{L} = \frac{1}{2} m v^2 - U \quad - (2)$$

In gravitational theory it is assumed that U is a function only of r:

$$U = U(r) \quad - (3)$$

In these equations m is a mass that orbits a mass M, and the velocity squared is:

$$v^2 = \dot{r}^2 + r^2 \dot{\beta}^2 \quad - (4)$$

where:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \quad - (5)$$

The lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) - U(r) \quad - (6)$$

The two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad - (7)$$

and

2)

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad - (8)$$

Eq. (7) implies:

$$m(\ddot{r} - r\dot{\beta}^2) = -\frac{\partial U(r)}{\partial r} = F(r) \quad - (9)$$

and eq. (8) implies:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = mr^2 \frac{d\beta}{dt} = mr^2\dot{\beta}$$

$$= \text{constant} \quad - (10)$$

where L is the magnitude of the total angular momentum.

Denote:

$$u = \frac{1}{r} \quad - (11)$$

Then:

$$\frac{du}{d\beta} = -\frac{1}{r^2}\frac{dr}{d\beta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\beta} \quad - (12)$$

$$= -\frac{m}{L}\frac{dr}{dt}.$$

Therefore:

$$\frac{d^2u}{d\beta^2} = \frac{d}{d\beta}\left(-\frac{m}{L}\frac{dr}{dt}\right)$$

$$= \frac{d}{dt}\left(-\frac{m}{L}\frac{dr}{dt}\right)\frac{dt}{d\beta}$$

$$= -\frac{m}{L}\frac{dt}{d\beta}\frac{d^2r}{dt^2} \quad -(13)$$

Therefore:

$$\ddot{r} = \frac{d^2r}{dt^2} = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\beta^2} \quad -(14)$$

$$r\dot{\beta}^2 = \frac{L^2}{m^2}u^3 \quad -(15)$$

Fm eqs. (9), (14) and (15):

$$\frac{d^2u}{d\beta^2} + u = -\frac{m}{L^2u^2}F(u) \quad -(16)$$

i.e.

$$\boxed{F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = -\frac{\partial U(r)}{\partial r}} \quad -(17)$$

This is the beta Binet equation for 3D orbits.

4) It was originally derived in the nineteenth century for 2D orbits, where:

$$F(r) = -\frac{L_Z^2}{mr^2}\left(\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(18)$$

$$= -\frac{\partial U(r)}{\partial r}$$

The Binet equation is useful for deriving the force law $F(r)$ needed for an orbit that has been measured experimentally. It was derived in astronomy for this purpose, using 2D orbital theory. For example if the orbit is a Keplerian ellipse in 2D:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(19)$$

then:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right) \quad -(20)$$

and

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\phi. \quad -(21)$$

So $$F(r) = \frac{L_Z^2}{mr^2}\frac{\epsilon}{\alpha}\cos\phi \quad -(22)$$

where:

$$\alpha = \frac{L_Z^2}{mk} \quad -(23)$$

and

$$\cos\phi = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(24)$$

So
$$F(r) = \frac{k}{r^2}\left(\frac{\alpha}{r} - 1\right) - \frac{L_Z^2}{mr^3}$$
$$= -\frac{k}{r^2} + \frac{k\alpha}{r^3} - \frac{L_Z^2}{mr^3}$$
$$= -\frac{k}{r^2} + \frac{L_Z^2}{mr^3} - \frac{L_Z^2}{mr^3}$$
$$= -\frac{k}{r^2} \quad -(25)$$

The force law needed for the orbit (19) is the inverse square law of attraction:

$$F(r) = -\frac{\partial U(r)}{\partial r} = -\frac{k}{r^2} \quad -(26)$$

where

$$k = mMG \quad -(27)$$

where M is the mass at one focus of the conical section attracting the mass m. Here G Newton's constant. In electrodynamics the one theory applies but:

$$k = \frac{e^2}{4\pi \epsilon_0} \quad -(28)$$

where e is the charge on the proton and ϵ_0 the vacuum permittivity.

The Leibniz equation of orbits is obtained from:

$$\boxed{m\ddot{r} = m\frac{d^2 r}{dt^2} = -\frac{L^2}{mr^2}\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right)} \quad -(29)$$

In 2D theory this becomes:

$$m\ddot{r} = m\frac{d^2 r}{dt^2} = -\frac{L_Z^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) \quad -(30)$$

If the orbit is the conic section (19) then:

$$\boxed{m\ddot{r} = m\frac{d^2 r}{dt^2} = -\frac{mMG}{r^2} + \frac{L_Z^2}{mr^3}} \quad -(31)$$

which is the 1689 Leibniz equation for 2D orbits. The centrifugal force is:

$$F_c = \frac{L_Z^2}{mr^3} \quad -(32)$$

). In 3D orbit theory the orbit is the beta ellipse

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(33)$$

and the 3D Leibniz equation is:

$$m\ddot{r} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} \quad -(34)$$

The L_Z of the 2D Leibniz equation is replaced by L of the 3D Leibniz equation. The inverse square law (26) is the same in 2D and 3D theory, but the centrifugal force is different in 2D and 3D theory. In 2D the centrifugal force is:

$$F_c = mr\dot{\phi}^2 = \frac{L_Z^2}{mr^3} \quad -(35)$$

but in 3D:

$$F_c = mr\dot{\beta}^2 = mr\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

$$= \frac{L^2}{mr^3} \quad -(36)$$

So $$\left(\frac{L}{L_Z}\right)^2 = \frac{\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)_{3D}}{\left(\dot{\phi}^2\right)_{2D}} \quad -(37)$$

8) where:

$$\dot{\phi}_{2D} = \frac{L_Z}{mr^2} \quad -(38)$$

$$\dot{\phi}_{3D} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(39)$$

$$\dot{\theta}_{3D} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(40)$$

Eqs. (37) to (40) are self consistent because they give:

$$\left(\frac{L}{L_Z}\right)^2 = \frac{1}{L_Z^2}\left(\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right) + \frac{L_Z^2}{\sin^2\theta}\right) \quad -(41)$$

From previous work:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi\right)^{1/2}} \quad -(42)$$

$$\sin\beta = -\frac{L\cos\theta}{(L^2 - L_Z^2)^{1/2}} \quad -(43)$$

for which:

$$\frac{\cos^2\theta}{1-y} + \frac{\cos^2\phi}{\cos^2\phi + y\sin^2\phi} = 1 \quad -(44)$$

$$= \cos^2\beta + \sin^2\beta$$

 where

$$y = \left(\frac{L_z}{L}\right)^2 \quad -(45)$$

This gives the quadratic equation:

$$y^2 \sin^2\phi + y \sin^2\phi\left(\cos^2\theta - 1\right) + \cos^2\phi \cos^2\theta = 0$$

$$-(46)$$

which reduces to:

$$y = 1 \quad -(47)$$

when

$$\theta = \frac{\pi}{2}, \quad \cos\theta = 0 \quad -(48).$$

So:

$$y = \frac{1}{2a}\left(-b \pm \left(b^2 - 4ac\right)^{1/2}\right) -(49)$$

$$-(50)$$

where

$$a = \sin^2\phi$$

$$b = \sin^2\phi\left(\cos^2\theta - 1\right) \quad -(51)$$

$$c = \cos^2\phi \cos^2\theta \quad -(52)$$

Therefore:

$$y = \left(\frac{L_z}{L}\right)^2 = \frac{1}{2}\left[1 - \cos^2\theta \pm \left(\sin^2\phi\left(\cos^2\theta - 1\right)^2 - 4\cos^2\phi\cos^2\theta\right)^{1/2}\right]$$

$$-(53)$$

Therefore $\frac{L_z}{L}$ may be expressed in terms of the

), coordinates θ and ϕ of the spherical polar system.

In order for the square root in eq. (53) to be real valued:

$$\sin^2\phi\left(\cos^2\theta - 1\right)^2 = \sin^2\phi\left(1 - \cos^2\theta\right)^2 \geq 4\cos^2\phi\cos^2\theta \quad -(54)$$

i.e. $$1 - \cos^2\theta \geq \frac{2\cos\phi\cos\theta}{\sin\phi} \quad -(55)$$

Since $$1 - \cos^2\theta \leq 1 \quad -(56)$$

then: $$\frac{2\cos\phi\cos\theta}{\sin\theta} \leq 1 \quad -(57)$$

i.e $$2\cos\phi\cos\theta \leq \sin\theta \quad -(58)$$

or $$\boxed{\tan\theta \geq 2\cos\phi} \quad -(59)$$

Finally, $(L_Z / L)^2$ must be positive valued.

So these conditions give the restriction on the range of θ and ϕ. With these restrictions, $(L_Z / L)^2$ can be graphed as a function of θ and ϕ
