274(b): Cartesian Format of the Beta Ellipse

This is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(1)$$

where

$$x = a\epsilon + r\cos\beta \quad -(2)$$

$$y = r\sin\beta \quad -(3)$$

these equations correspond to:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(4)$$

where:

$$\alpha = a(1-\epsilon^2) \quad -(5)$$

$$\epsilon^2 = 1 - \frac{b^2}{a^2} \quad -(6)$$

where a and b are the major and minor semi axes of the ellipse.

Proof

Eq. (1) is:

$$\frac{(c + r\cos\beta)^2}{a^2} + \frac{r^2\sin^2\beta}{b^2} = 1 \quad -(7)$$

where

$$c := a\epsilon \quad -(8)$$

So:

2)

$$b^2\left(c^2+2cr\cos\beta+r^2\cos^2\beta\right)+a^2r^2\sin^2\beta=a^2b^2 \quad -(9)$$

i.e.

$$b^2c^2+2rcb^2\cos\beta+b^2r^2\cos^2\beta+a^2r^2-a^2r^2\cos^2\beta$$
$$=a^2b^2 \quad -(10)$$

Now use: $\quad b^2=a^2\left(1-\epsilon^2\right) \quad -(11)$

so:

$$a^2\left(1-\epsilon^2\right)a^2\epsilon^2+2a\epsilon a^2\left(1-\epsilon^2\right)r\cos\beta$$
$$+a^2\left(1-\epsilon^2\right)r^2\cos^2\beta+a^2r^2-a^2r^2\cos^2\beta$$
$$=a^2\left(a^2\left(1-\epsilon^2\right)\right). \quad -(12)$$

Therefore:

$$a^2\left(1-\epsilon^2\right)\epsilon^2+2a\epsilon\left(1-\epsilon^2\right)r\cos\beta+\left(1-\epsilon^2\right)r^2\cos^2\beta$$
$$+r^2-r^2\cos^2\beta=a^2\left(1-\epsilon^2\right) \quad -(13)$$

and:

$$r^2=r^2\cos^2\beta+a^2\left(1-\epsilon^2\right)-a^2\left(1-\epsilon^2\right)\epsilon^2$$
$$-2a\epsilon\left(1-\epsilon^2\right)r\cos\beta-\left(1-\epsilon^2\right)r^2\cos^2\beta \quad -(14)$$
$$=\epsilon^2r^2\cos^2\beta-2a\epsilon\left(1-\epsilon^2\right)r\cos\beta$$
$$+a^2\left(1-\epsilon^2+\epsilon^2\left(1-\epsilon^2\right)\right)$$

Therefore:

$$r^2 = \epsilon^2 r^2 \cos^2\beta - 2a\epsilon(1-\epsilon^2) r\cos\beta + a^2(1-\epsilon^2)^2 \quad -(15)$$

$$= \left(\epsilon r\cos\beta - a(1-\epsilon^2)\right)^2$$

Therefore:

$$r = \pm\left(\epsilon r\cos\beta - a(1-\epsilon^2)\right) \quad -(16)$$

Taking the negative root:

$$r = -\left(\epsilon r\cos\beta - a(1-\epsilon^2)\right) \quad -(17)$$

i.e.

$$\boxed{r = \frac{\alpha}{1+\epsilon\cos\beta}} \quad -(18)$$

<u>Q.E.D</u>, where:

$$\alpha = a(1-\epsilon^2) \quad -(19)$$

is the half right latitude.

Taking the positive root:

$$r = \epsilon r\cos\beta - \alpha \quad -(20)$$

so:

$$r(\epsilon\cos\beta - 1) = \alpha$$

$$\boxed{r = \frac{\alpha}{\epsilon\cos\beta - 1}} \quad -(21)$$

t)

Therefore the beta ellipse is defined by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(22)$$

where:

$$x = a\epsilon + r\cos\beta \quad -(23)$$

$$y = r\sin\beta \quad -(24)$$

It is transformed into a 3D ϕ ellipse

using:

$$\cos^2\beta = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi} \quad -(25)$$

$$\sin^2\beta = \frac{\cos^2\theta}{1 - \left(\frac{L_z}{L}\right)^2} \quad -(26)$$