274(3): Expression for Precessing Ellipse in Terms of 3D Orbital Theory.

This expression is obtained by starting with:

$$\cos^2\beta = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi} \quad -(1)$$

with:

$$\sin\phi = \frac{Y}{r}, \quad \cos\phi = \frac{X}{r} \quad -(2)$$

$$\tan\phi = \frac{Y}{X} \quad -(3)$$

So

$$\cos^2\beta = \frac{X^2}{X^2 + \left(\frac{L_Z}{L}\right)^2 Y^2} \quad -(4)$$

First note that:

$$\sin^2\beta = \left(\frac{L^2}{L^2 - L_Z^2}\right)\left(\frac{Z}{r}\right)^2 \quad -(5)$$

$$= 1 - \cos^2\beta$$

So

$$\left(\frac{Z}{r}\right)^2 = \left(1 - \left(\frac{L_Z}{L}\right)^2\right)\left(1 - \frac{X^2}{X^2 + \left(\frac{L_Z}{L}\right)^2 Y^2}\right) \quad -(6)$$

where

$$r^2 = X^2 + Y^2 + Z^2 \quad -(7)$$

2) Therefore Z^2 can be expressed in terms of X^2 and Y^2 for an inputted $(L_Z / L)^2$. We can consider two special cases as follows.

1) When:

$$L = L_Z \quad -(8)$$

then:

$$Z = 0 \quad -(9)$$

This gives the planar or two dimensional orbit.

2) For a spherical orbit:

$$\left(\frac{Z}{r}\right)^2 = \frac{1}{3} \quad -(10)$$

and

$$X^2 = Y^2 = Z^2 \quad -(11)$$

so:

$$\frac{1}{3} = \left(1 - \left(\frac{L_Z}{L}\right)^2\right)\left(1 - \frac{1}{1 + \left(\frac{L_Z}{L}\right)^2}\right) \quad -(12)$$

This equation can be solved for $(L_Z / L)^2$ needed for a spherical orbit.

The general orbital equation is:

$$\boxed{\frac{Z^2}{X^2 + Y^2 + Z^2} = \left(1 - \left(\frac{L_Z}{L}\right)^2\right)\left(1 - \frac{X^2}{X^2 + \left(\frac{L_Z}{L}\right)^2 Y^2}\right) \quad -(13)}$$

3) Eq. (13) can be solved for $(L_Z / L)^2$ in terms of X, Y and Z as follows. Let:

$$B = \left(\frac{Z}{r}\right)^2, \quad y = \left(\frac{L_Z}{L}\right)^2 \quad -(14)$$

then

$$(1-y)\left(1 - \frac{X^2}{X^2 + yY^2}\right) = B \quad -(15)$$

and

$$\boxed{y^2 + (B-1)y + \left(\frac{X}{Y}\right)^2 B = 0} \quad -(16)$$

This is the same as eq. (26) of note 274(2), so the method is self consistent. Therefore:

$$y = \frac{1}{2}\left[1 - B \pm \left((1-B)^2 - 4B\left(\frac{X}{Y}\right)^2\right)^{1/2}\right] \quad -(17)$$

$$= \left(\frac{L_Z}{L}\right)^2$$

Since $y \rightarrow 1$ for $B \rightarrow 0$, the positive root is needed in eq. (17):

$$y = \frac{1}{2}\left[1 - B + \left((1-B)^2 - 4B\left(\frac{X}{Y}\right)^2\right)^{1/2}\right] \quad -(18)$$

In order for y to be real valued and positive:

) In order for this to reduce correctly:

$$(1-B)^2 > 4B\left(\frac{X}{Y}\right)^2 \quad -(19)$$

and

$$(1-B)^2 \neq 4B\left(\frac{X}{Y}\right)^2 \quad -(20)$$

so

$$\boxed{\frac{Z}{r}\left(\frac{Z}{r} + 2\frac{X}{Y}\right) < 1} \quad -(21)$$

Eq. (21) is always valid for a 2D orbit because:

$$Z = 0 \quad -(22)$$

A precessional orbit can be constructed if it is assumed that:

$$\beta = x\phi \quad -(23)$$

Then the beta ellipse reduces to:

$$r = \frac{\alpha}{1+\epsilon\cos(x\phi)} \quad -(24)$$

From eqs. (1) and (23):

$$\cos^2(x\phi) = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_Z}{L}\right)^2\sin^2\phi} \quad -(25)$$

so:

$$\left(\frac{L_Z}{L}\right)^2 = \frac{\cos^2\phi}{\sin^2\phi}\left(\frac{1}{\cos^2(x\phi)} - 1\right) \quad -(26)$$

By observation:

$$x = 1 + \frac{3MG}{c^2\alpha} \quad -(27)$$

So the Seta ellipse reduces to the precessing ellipse in a plane, eq. (24) provided that L_Z/L is given by eq. (26). The range of validity of eq. (26) is restricted to:

$$\left(\frac{L_Z}{L}\right)^2 \leq 1 \quad -(28)$$

Therefore in order to obtain a constant x as observed in planetary orbits, L_Z/L must depend on ϕ in the Seta ellipse of 3D orbits.

To an excellent approximation:

$$x \sim 1 \quad -(29)$$

so

$$\left(\frac{L_Z}{L}\right)^2 = \frac{\cos^2\phi}{\sin^2\phi}\left(\frac{1-\cos^2(x\phi)}{\cos^2(x\phi)}\right)$$

$$\sim \frac{\cos^2\phi\,(1-\cos^2\phi)}{\sin^2\phi\,\cos^2(x\phi)}$$

i.e. $$\boxed{\left(\frac{L_z}{L}\right)^2 \sim \frac{\cos^2\phi}{\cos^2(x\phi)}} \quad -(30)$$

with: $$\frac{L_z}{L} \rightarrow 1 \text{ as } x \rightarrow 1 \quad -(31)$$

Eq. (30) simplifies to:

$$\frac{L_z}{L} \sim \frac{\cos\phi}{\cos(x\phi)} \quad -(32)$$

or $$\cos(x\phi) = \left(\frac{L}{L_z}\right)\cos\phi \quad -(33)$$

and $$\cos\phi = \frac{L_z}{L}\cos(x\phi) \quad -(34)$$

With reference to Fig (1) the range of eq (34) is

$$\frac{3\pi}{2} < \phi < 2\pi \quad -(35)$$

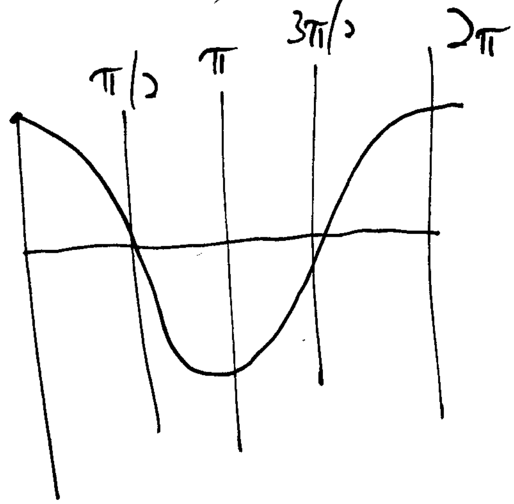

Fig (1)

because in this range:

$$\cos(x\phi) > \cos\phi \quad -(36)$$

and $$\frac{L_z}{L} < 1 \quad -(37)$$