74(7): Cartesian Relations in the Beta Ellipse

The Cartesian representation of the beta ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(1)$$

where

$$x = a\epsilon + r\cos\beta \quad -(2)$$

$$y = r\sin\beta \quad -(3)$$

and where a and b are the semi major and semi minor axes. Eqs. (1) to (3) correspond to:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(4)$$

$$\alpha = a(1-\epsilon^2) \quad -(5)$$

$$\epsilon^2 = 1 - \frac{b^2}{a^2} \quad -(6)$$

and to the Hamiltonian:

$$H = \frac{1}{2}mv^2 - \frac{k}{r} \quad -(7)$$

where

$$v^2 = \dot{r}^2 + \dot{\beta}^2 r^2 \quad -(8)$$

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(9)$$

and the Lagrangian:

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{r}. \quad -(10)$$

The possible Euler Lagrange equations are:

2) $$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(11)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) \quad -(12)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(13)$$

and $$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(14)$$

Eqs. (10) to (14) give:

$$m\ddot{r} = mr\dot{\beta}^2 - \frac{L^2}{mr^2} \quad -(15)$$

$$\frac{d}{dt}\left(mr^2\dot{\beta}\right) = 0 \quad -(16)$$

$$\frac{d}{dt}\left(mr^2 \sin^2\theta\, \dot{\phi}\right) = 0 \quad -(17)$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 2\sin\theta\cos\theta\,\dot{\phi}^2 \quad -(18)$$

From basic considerations as in UFT 269:

$$L_Z = mr^2 \sin^2\theta\,\dot{\phi} \quad -(19)$$

$$L^2 = m^2 r^4\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \quad -(20)$$

so eq. (16) is: $$\frac{dL}{dt} = 0 \quad -(21)$$

3) and eq. (17) is:

$$\frac{dL_Z}{dt} = 0 \quad -(22)$$

Eq. (18) shows that $mr^2\dot{\theta}$ is not a constant of motion, but eqs. (16) and (17) mean that L and L_Z are constants of motion.

From eqs. (19) and (20):

$$\frac{d\theta}{dt} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(23)$$

and

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(24)$$

$$\frac{d\phi}{dt} = \frac{L_Z}{mr^2\sin^2\theta} \quad -(25)$$

Eqs. (23) to (25) are true for a kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right) \quad -(26)$$

and for any potential energy U.

It follows that:

4)

$$\frac{d\beta}{d\theta} = \frac{L}{\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2}} \quad - (27)$$

and $$\frac{d\beta}{d\phi} = \left(\frac{L}{L_Z}\right) \sin^2\theta \quad - (28)$$

Eqs. (27) and (28) give the relation between β and θ and β and ϕ.

Therefore:

$$\beta = \int \frac{L\, d\theta}{\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2}} = -\sin^{-1}\left(\frac{L\cos\theta}{\left(L^2 - L_Z^2\right)^{1/2}}\right) \quad - (29)$$

and $$\sin\beta = -\frac{L\cos\theta}{\left(L^2 - L_Z^2\right)^{1/2}}. \quad - (30)$$

It follows that:

$$\cos^2\beta = 1 - \sin^2\beta = 1 - \frac{L^2\cos^2\theta}{\left(L^2 - L_Z^2\right)} \quad - (31)$$

From Eq. (28):

$$\beta = \int \frac{L}{L_z} \sin^2\theta \, d\phi \quad -(32)$$

and

$$\phi = \frac{L_z}{L} \int \frac{d\beta}{\sin^2\theta}. \quad -(33)$$

From Eq. (30):

$$\left(\frac{L^2}{L^2 - L_z^2}\right) \cos^2\theta = \sin^2\beta \quad -(34)$$

$$\cos^2\theta = \left(1 - \left(\frac{L_z}{L}\right)^2\right) \sin^2\beta \quad -(35)$$

So

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(1 - \left(\frac{L_z}{L}\right)^2\right) \sin^2\beta. \quad -(36)$$

and

Therefore:

$$\phi = \frac{L_z}{L} \int \left(1 - \left(1 - \left(\frac{L_z}{L}\right)^2\right) \sin^2\beta\right)^{-1} d\beta \quad -(37)$$

$$= \tan^{-1}\left(\frac{L_z}{L} \tan\beta\right)$$

Therefore:

$$\tan\phi = \frac{L_z}{L} \tan\beta \quad -(38)$$

). It follows that:

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\left(1-\cos^2\beta\right)^{1/2}}{\cos\beta} = \frac{L}{L_Z}\tan\phi \qquad -(39)$$

so:

$$1-\cos^2\beta = \frac{L^2}{L_Z^2}\tan^2\phi\cos^2\beta \qquad -(40)$$

and

$$\cos^2\beta = \frac{1}{1+\left(\frac{L}{L_Z}\right)^2\tan^2\phi} \qquad -(41)$$

$$= \frac{\cos^2\phi}{\sin^2\phi+\left(\frac{L}{L_Z}\right)^2\cos^2\phi}$$

Therefore:

$$X = a\epsilon + \frac{r\cos\phi}{\left(\sin^2\phi+\left(\frac{L}{L_Z}\right)^2\cos^2\phi\right)^{1/2}} \qquad -(42)$$

$$Y = \frac{r\cos\theta}{\left(1-\left(\frac{L_Z}{L}\right)^2\right)^{1/2}} = \frac{Z}{\left(1-\left(\frac{L_Z}{L}\right)^2\right)^{1/2}} \qquad -(43)$$

7) Eq. (43) shows clearly that:

$$\boxed{Z_1 = \left(1 - \left(\frac{L_Z}{L}\right)^2\right)^{1/2} Y} \quad -(44)$$

$$\rightarrow 0$$

as

$$L \rightarrow L_Z \quad -(45)$$

This is an indication that the orbit is three dimensional.

The equation linking Z_1 and X is found from eqs. (1) and (44):

$$\boxed{\frac{X^2}{a^2} + \frac{Z_1^2}{b^2\left(1 - \left(\frac{L_Z}{L}\right)^2\right)} = 1} \quad -(46)$$

where

$$\boxed{\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1} \quad -(47)$$