274(4): Theory of Three Dimensional Galaxies

In two dimensional whirlpool galaxies the orbit of a star around the central mass is the hyperbolic spiral:

$$r = \frac{r_0}{\phi} \quad -(1)$$

The transition from 2D to 3D theory is given by

$$\boxed{\phi \rightarrow \beta} \quad -(2)$$

where

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(3)$$

so Eq. (1) becomes:

$$r = \frac{r_0}{\beta} \quad -(4)$$

The three dimensional Binet equation is:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(5)$$

where

$$m\ddot{r} = -\frac{L^2}{mr^2}\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) \quad -(6)$$

From eq. (4):

$$\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) = 0 \quad -(7)$$

So

$$m\ddot{r} = 0 \quad -(8)$$

and

$$F(r) = -\frac{L^2}{mr^3} \quad -(9)$$

The force law needed for the beta spiral (2) is the inverse cube law (9). The potential is defined by:

$$F(r) = -\frac{\partial U(r)}{\partial r} = -\frac{L^2}{mr^3} \quad -(10)$$

so

$$U(r) = -\frac{L^2}{2mr^2} \quad -(11)$$

The hamiltonian of the 3D galaxy is:

$$H = \frac{1}{2}mv^2 + U(r) \quad -(12)$$

$$= \frac{1}{2}mv^2 - \frac{L^2}{2mr^2}$$

where

$$v^2 = \dot{r}^2 + r^2\dot{\beta}^2$$

$$= \dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \quad -(13)$$

The lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) + \frac{L^2}{2mr^2} \quad -(14)$$

and the two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(15)$$

and

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) \quad -(16)$$

Eq. (15) gives:

$$m\ddot{r} = mr\dot{\beta}^2 - \frac{L^2}{mr^3} \quad -(17)$$

and Eq. (16) gives:

$$\frac{d}{dt}\left(mr^2\dot{\beta}\right) = 0 \quad -(18)$$

Therefore the conserved angular momentum is

$$L = mr^2\dot{\beta} \quad -(19)$$

so

$$\dot{\beta} = \frac{L}{mr^2} \quad -(20)$$

From eqs. (17) and (20):

4)

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{L^2}{mr^3} = 0 \qquad -(21)$$

which is eq. (8) from the Binet equation QED.

The three dimensional orbit corresponding to the hamiltonian (12) is:

$$\boxed{r = \frac{r_0}{\beta}} \qquad -(22)$$

As in previous work the relation between β and ϕ and θ is given by:

$$\cos^2\beta = \frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi} \qquad -(23)$$

and

$$\sin^2\beta = \frac{L^2}{L^2 - L_Z^2} \cos^2\theta \qquad -(24)$$

where

$$\cos\theta = \frac{Z}{r} \qquad -(25)$$

$$\sin\phi = \frac{Y}{r} \qquad -(26)$$

$$\cos\phi = \frac{X}{r} \qquad -(27)$$

Therefore:

$$\beta = \cos^{-1}\left(\frac{x^2}{x^2 + \left(\frac{L_z}{L}\right)^2 y^2}\right) \quad -(28)$$

$$= \sin^{-1}\left(\left(\frac{1}{1-\left(\frac{L_z}{L}\right)^2}\right)\left(\frac{Z}{r}\right)^2\right)$$

Therefore the orbit (22) can be plotted in terms of X and Y, or in terms of Z, with input parameter L_z / L. Here:

$$r^2 = X^2 + Y^2 + Z^2 \quad -(29)$$

and:

$$\left(\frac{Z}{r}\right)^2 = \frac{Z^2}{X^2+Y^2+Z^2} = \left(1-\left(\frac{L_z}{L}\right)^2\right)\left(1 - \frac{X^2}{X^2 + \left(\frac{L_z}{L}\right)^2 Y^2}\right) \quad -(\cancel{25}\ 30)$$

Note that Eqs. (23) to (30) are true for any force law.