274(2): Development of 3D Orbital Theory in Cartesian Coordinates

Consider the hamiltonian in spherical polar coordinates:

$$H = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) \right) - \frac{k}{r} \quad -(1)$$

in the notation of previous work. Then the orbit is:

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(2)$$

where:

$$\cos \beta = \frac{\cos \phi}{\left(\cos^2 \phi + \left(\frac{L_Z}{L} \right)^2 \sin^2 \phi \right)^{1/2}} \quad -(3)$$

$$\sin \beta = \frac{-L \cos \theta}{\left(L^2 - L_Z^2 \right)^{1/2}} \quad -(4)$$

As in the previous note:

$$\left(\frac{L_Z}{L} \right)^2 = \frac{1}{2} \left(1 - \cos^2 \theta \pm \left(\sin^2 \phi \left(\cos^2 \theta - 1 \right)^2 - 4 \cos^2 \phi \cos^2 \theta \right)^{1/2} \right) \quad -(5)$$

In order to transform into Cartesian coordinates:

$$X = r \sin \theta \cos \phi \quad -(6)$$

$$Y = r \sin \theta \sin \phi \quad -(7)$$

$$Z = r \cos \theta \quad -(8)$$

$$r = (X^2 + Y^2 + Z^2)^{1/2} \quad -(9)$$

$$\theta = \cos^{-1}\frac{Z}{r} \quad -(10)$$

Here θ is restricted to the principle value range:

$$0 \leqslant \theta \leqslant \pi \quad -(11)$$

and

$$\phi = \tan^{-1}\frac{Y}{X} \quad -(12)$$

$$0 \leqslant \phi \leqslant 2\pi \quad -(13)$$

So:

$$\cos\theta = \frac{Z}{r} \ ; \ \tan\phi = \frac{Y}{X} \quad -(14)$$

From previous work:

$$\tan\beta = \frac{L}{L_Z}\tan\phi = \frac{L}{L_Z}\frac{Y}{X} \quad -(15)$$

As in note 272(2):

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{(1-\cos^2\beta)^{1/2}}{\cos\beta} = \frac{L_Z}{L}\tan\phi \quad -(16)$$

So:

$$1 - \cos^2\beta = \left(\frac{L_Z}{L}\right)^2 \cos^2\beta \tan^2\phi \quad -(17)$$

and

$$\cos^2\beta = \frac{1}{1 + \left(\frac{L_Z}{L}\right)^2 \tan^2\phi} \quad -(18)$$

3) i.e.

$$\cos^2\beta = \frac{1}{1+\left(\frac{L_z}{L}\right)^2\left(\frac{y}{x}\right)^2} \quad -(19)$$

$$\sin^2\beta = \frac{L^2}{L^2-L_z^2}\left(\frac{z}{r}\right)^2$$

$$= \frac{1}{1-\left(\frac{L_z}{L}\right)^2}\left(\frac{z}{r}\right)^2 \quad -(20)$$

Adding Eqs. (19) and (20):

$$\frac{1}{1+\left(\frac{L_z}{L}\right)^2\left(\frac{y}{x}\right)^2} + \frac{1}{\left(1-\left(\frac{L_z}{L}\right)^2\right)\left(\frac{r}{z}\right)^2} = 1 \quad -(21)$$

So $$\left(1-\left(\frac{L_z}{L}\right)^2\right)\left(\frac{r}{z}\right)^2 + 1 + \left(\frac{L_z}{L}\right)^2\left(\frac{y}{x}\right)^2$$

$$= \left(1-\left(\frac{L_z}{L}\right)^2\right)\left(\frac{r}{z}\right)^2\left(1+\left(\frac{L_z}{L}\right)^2\left(\frac{y}{x}\right)^2\right) \quad -(22)$$

Let $$y = \left(\frac{L_z}{L}\right)^2 \quad -(23)$$

Then:

$$\left(1-y\right)\left(\frac{r}{Z}\right)^2 + 1 + \left(\frac{Y}{X}\right)^2 y$$

$$= \left(\frac{r}{Z}\right)^2\left(1-y\right)\left(1+y\left(\frac{Y}{X}\right)^2\right) \quad -(23)$$

So:

$$1 + \left(\frac{Y}{X}\right)^2 y = y\left(1-y\right)\left(\frac{r}{Z}\right)^2\left(\frac{Y}{X}\right)^2$$

$$= y\left(\frac{r}{Z}\right)^2\left(\frac{Y}{X}\right)^2 - y^2\left(\frac{r}{Z}\right)^2\left(\frac{Y}{X}\right)^2 \quad -(24)$$

i.e.

$$y^2\left(\frac{r}{Z}\right)^2\left(\frac{Y}{X}\right)^2 + y\left(\frac{Y}{X}\right)^2\left(1-\left(\frac{r}{Z}\right)^2\right) + 1 = 0. \quad -(25)$$

or

$$y^2 - \left(1-\left(\frac{Z}{r}\right)^2\right)y + \left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2 = 0 \quad -(26)$$

If

$$Z = 0 \quad -(27)$$

Eq. (26) reduces correctly to:

$$y = 1. \quad -(28)$$

Therefore:

$$y = \frac{1}{2a}\left(-b \pm \left(b^2 - 4ac\right)^{1/2}\right) \quad -(29)$$

where

$$a = 1 \quad -(30)$$

$$b = -\left(1-\left(\frac{Z}{r}\right)^2\right); \quad -(31)$$

$$c = \left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2 \quad -(32)$$

So:

$$y = \frac{1}{2}\left[1-\left(\frac{Z}{r}\right)^2 \pm \left(\left(1-\left(\frac{Z}{r}\right)^2\right)^2 - 4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2\right)^{1/2}\right] \quad -(33)$$

In order for $y \to 1$ for $Z = 0$ the positive root is needed, so:

$$y = \frac{1}{2}\left[1-\left(\frac{Z}{r}\right)^2 + \left(\left(1-\left(\frac{Z}{r}\right)^2\right)^2 - 4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2\right)^{1/2}\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{Z}{r}\right)^2\left(1 + \left(1 - \frac{4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2}{\left(1-\left(\frac{Z}{r}\right)^2\right)^2}\right)^{1/2}\right)\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{Z}{r}\right)^2\left(1 + \left(\frac{\left(1-\left(\frac{Z}{r}\right)^2\right)^2 - 4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2}{\left(1-\left(\frac{Z}{r}\right)^2\right)^2}\right)^{1/2}\right)\right]$$

$$= \frac{1}{2}\left[1-\left(\frac{Z}{r}\right)^2 + \left(\left(1-\left(\frac{Z}{r}\right)^2\right)^2 - 4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2\right)^{1/2}\right] \quad -(34)$$

6)

Result

The orbit corresponding to the three dimensional hamiltonian (1) is:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(35)$$

where

$$\cos^2\beta = \frac{1}{1+\left(\frac{L_Z}{L}\right)^2\left(\frac{Y}{X}\right)^2} \quad -(36)$$

and

$$\left(\frac{L_Z}{L}\right)^2 = \frac{1}{2}\left[1-\left(\frac{Z}{r}\right)^2+\left(\left(1-\left(\frac{Z}{r}\right)^2\right)^2-4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2\right)^{1/2}\right] \quad -(37)$$

where

$$r^2 = X^2+Y^2+Z^2 \quad -(38)$$

The planar limit is defined by:

$$Z = 0 \quad -(39)$$

giving:

$$L = L_Z \quad -(40)$$

and

$$\cos\beta = \frac{X}{(X^2+Y^2)^{1/2}} \quad -(41)$$

$$= \cos\phi$$

7) So the planar orbit is:

$$r = \frac{\alpha}{1 + \epsilon \cos\phi} \quad -(42)$$

Q.E.D.

Range of Validity

In order for $\left(\frac{L_Z}{L}\right)^2$ to be real valued and positive:

$$1 - \left(\frac{Z}{r}\right)^2 \geq 2\left(\frac{Z}{r}\right)\left(\frac{X}{Y}\right) \quad -(43)$$

so

$$\left(\frac{Z}{r}\right)\left(\left(\frac{Z}{r}\right) + 2\left(\frac{X}{Y}\right)\right) \leq 1 \quad -(44)$$

where

$$r = (X^2 + Y^2 + Z^2)^{1/2} \quad -(45)$$

Examples

1) For a planar orbit:

$$Z = 0 \quad -(46)$$

and

$$r = \frac{\alpha}{1 + \epsilon \cos\phi} \quad -(47)$$

2) For a spherical orbit:

$$X = Y = Z \quad -(48)$$

8)

In general:

$$\left(\frac{L_Z}{L}\right)^2 = A + B \quad -(49)$$

where:

$$A = \frac{1}{2}\left(1 - \left(\frac{Z}{r}\right)^2\right) \quad -(50)$$

and

$$B = \frac{1}{2}\left(\left(1 - \left(\frac{Z}{r}\right)^2\right)^2 - 4\left(\frac{Z}{r}\right)^2\left(\frac{X}{Y}\right)^2\right)^{1/2}$$

$$-(51)$$

In general B is complex valued:

$$B = B' + iB'' \quad -(52)$$

1) If

$$1 - \left(\frac{Z}{r}\right)^2 > 2\left(\frac{Z}{r}\right)\left(\frac{X}{Y}\right) \quad -(53)$$

then

$$B = B' \quad -(54)$$

is real valued

2) If

$$1 - \left(\frac{Z}{r}\right)^2 = 2\left(\frac{Z}{r}\right)\left(\frac{X}{Y}\right) \quad -(55)$$

then

$$B = 0 \quad -(56)$$

3) If

$$1 - \left(\frac{Z}{r}\right)^2 < 2\left(\frac{Z}{r}\right)\left(\frac{X}{Y}\right) \quad -(57)$$

then

$$B = iB'' \quad -(58)$$

is imaginary valued.

In the special case of the orbit defined by Eq. (55):

$$\left(\frac{L_Z}{L}\right)^2 = \frac{1}{2}\left(1 - \left(\frac{Z}{r}\right)^2\right) \quad -(59)$$

and

$$\left(\frac{Z}{r}\right)^2 + 2\left(\frac{X}{Y}\right)\left(\frac{Z}{r}\right) - 1 = 0 \quad -(60)$$

so:

$$\frac{Z}{r} = \frac{1}{2}\left(-\frac{2X}{Y} \pm \left(4\frac{X^2}{Y^2} + 4\right)^{1/2}\right) \quad -(61)$$

$$\frac{Z}{r} = \pm\left(\frac{X^2}{Y^2} + 1\right)^{1/2} - \frac{X}{Y} \quad -(62)$$

where

$$r = \left(X^2 + Y^2 + Z^2\right)^{1/2} \quad -(63)$$

For this type of orbit Z can be evaluated in terms of X and Y.

<u>Conclusion</u>

There are many possibilities of building up three dimensional orbits in terms of X, Y and Z from eqs. (35) to (38). In general $r = r(X, Y, Z)$.