271(2): Velocity in Cartesian and Plane Polar

In Cartesian:

$$\underline{v} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} \quad -(1)$$

where

$$x = r\cos\theta, \quad y = r\sin\theta \quad -(2)$$

so

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad -(3)$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \quad -(4)$$

The unit vectors are related by:

$$\underline{i} = \underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta \quad -(5)$$

$$\underline{j} = \underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta \quad -(6)$$

so

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(7)$$

with:

$$\underline{e}_r \times \underline{e}_\theta = \underline{k} \quad -(8)$$

$$\underline{k} \times \underline{e}_r = \underline{e}_\theta \quad -(9)$$

$$\underline{e}_\theta \times \underline{k} = \underline{e}_r \quad -(10)$$

So:

$$\underline{v} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(11)$$

This relation is true, but in Cartesians:

$$\underset{\sim}{r} = x\,\underline{i} + y\,\underline{j} \quad -(12)$$

and in plane polars:

2)

$$\underset{\sim}{r} = r\,\underline{e}_r \quad - (13)$$

Differentiating eq. (13) gives:

$$\underline{v} = \frac{d\underline{r}}{dt} \quad - (14)$$

but differentiating eq. (13) gives:

$$\underline{v} = \frac{dr}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad - (15)$$

Here:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad - (16)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad - (17)$$

In order to introduce rotational motion, eqs. (2) are needed, so the angular velocity enters into the analysis through eqs. (3) and (4). In an inertial Newtonian analysis, only equation (1) is used, and the angular velocity does not enter into the analysis. So if a particle travels along $\underline{i}$, its velocity is:

$$\underline{v}_x = \frac{dx}{dt}\,\underline{i} \quad - (18)$$

and if it travels along $\underline{j}$ its velocity is:

$$\underline{v}_y = \frac{dy}{dt}\,\underline{j} \quad - (19)$$

<u>and the angular velocity does not appear</u>.

3) Newtonian motion is along a straight line, so the velocity is along a straight line.

In order to obtain an orbit the velocity must be transformed as follows:

$$\underline{v} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} \rightarrow \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(20)$$

and this introduces the angular velocity:

$$\omega = \frac{d\theta}{dt} \quad -(21)$$

and the term $\underline{\omega} \times \underline{r}$.

The transformation to spherical polar coordinates is:

$$\underline{v} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} + \dot{z}\,\underline{k}$$

$$\rightarrow \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(22)$$

and this introduces two angular velocities $\dot{\theta}$ and $\dot{\phi}$. In the initial Newtonian analysis the velocity is along a straight line in three dimensions and there are no angular velocities.