271(1): Comparison of the Lagrangian in 2-D Cartesian and Plane Polar Coordinates.

The two Lagrangians are:

$$\mathcal{L}_1 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{k}{r} \quad -(1)$$

and

$$\mathcal{L}_2 = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{k}{r} \quad -(2)$$

where

$$r = \left(x^2 + y^2 \right)^{1/2} \quad -(3)$$

In Cartesian:

$$\underline{v} = \dot{x} \underline{i} + \dot{y} \underline{j} = \frac{d}{dt} \left(x \underline{i} + y \underline{j} \right) \quad -(4)$$

Note carefully that:

$$\frac{d\underline{i}}{dt} = \frac{d\underline{j}}{dt} = \underline{0} \quad -(5)$$

In plane polar:

$$\dot{\underline{e}}_r = \dot{\theta} \underline{e}_\theta , \quad \dot{\underline{e}}_\theta = -\dot{\theta} \underline{e}_r \quad -(6)$$

so $\underline{e}_r$ and $\underline{e}_\theta$ depend on time. Therefore:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt} \left(r \underline{e}_r \right) = \dot{r} \underline{e}_r + r \dot{\underline{e}}_r = \dot{r} \underline{e}_r + r \dot{\theta} \underline{e}_\theta \quad -(7)$$

2) So the two Lagrangians (1) and (2) are not the same.

The reason is that:

$$\frac{d}{dt}(x\underline{i}) = \dot{x}\underline{i}, \quad \frac{d}{dt}(y\underline{j}) = \dot{y}\underline{j} \quad -(8)$$

but by the Leibniz Theorem:

$$\frac{d}{dt}(r\underline{e}_r) = \dot{r}\underline{e}_r + \boxed{r\dot{\underline{e}}_r} \quad -(9)$$

Using the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}_1}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \dot{x}} \quad -(10)$$

and

$$\frac{\partial \mathcal{L}_1}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \dot{y}} \quad -(11)$$

it is found that:

$$\ddot{x} + \left(\frac{mG}{r^3}\right)x = 0 \quad -(12)$$

$$\ddot{y} + \left(\frac{mG}{r^3}\right)y = 0 \quad -(13)$$

These are harmonic oscillator equations, not ellipses. Their solutions are:

$$x = e^{i\omega t}x_0 \quad -(14)$$

$$y = e^{i\omega t}y_0 \quad -(15)$$

$$\omega = \left(\frac{mG}{r^3}\right)^{1/2} \quad -(16)$$

3) They can be expressed as:

$$\frac{xx^*}{2x_0^2} + \frac{yy^*}{2y_0^2} = 1 \quad -(17)$$

which looks like an ellipse superficially, but eq. (17) is just:

$$\frac{1}{2} + \frac{1}{2} = 1 \quad -(18)$$

The true ellipse is given by the lagrangian (2) with:

$$\frac{\partial \mathcal{L}_2}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}_2}{\partial \dot{r}}\right) \quad -(19)$$

$$\frac{\partial \mathcal{L}_2}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}_2}{\partial \dot{\theta}}\right) \quad -(20)$$

These give:

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{k}{r^2} \quad -(21)$$

and

$$\dot{\theta} = \frac{L}{mr^2} \quad -(22)$$

so:

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{k}{r^2} \quad -(23)$$

which is the 1689 Leibniz equation.

Eqs (21) to (23) can be rewritten as:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad - (24)$$

where:

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad - (25)$$

Note carefully that eq. (7) is a covariant derivative:

$$\underline{v} = \frac{D\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \quad - (26)$$

where

$$\underline{\omega} \times \underline{r} = r\dot{\theta}\,\underline{e}_\theta \quad - (27)$$

Thus:

$$\underline{v} = \underbrace{\boxed{\frac{dr}{dt}\underline{e}_r}}_{\text{Newtonian}} + \underbrace{\boxed{\underline{\omega} \times \underline{r}}}_{\text{Non-Newtonian}} \quad - (28)$$

The angular velocity:

$$\underline{\omega} = \frac{d\theta}{dt}\,\underline{k} \quad - (29)$$

is a spin connection of Cartan geometry. We have:

$$\underline{\omega} \times \underline{r} = \dot{\theta}\,\underline{k} \times r\,\underline{e}_r \quad - (30)$$
$$= r\dot{\theta}\,\underline{e}_\theta$$