271(7): Evaluation of the Constraint Equations.

In previous notes the constraint equations were evaluated:

$$m\ddot{r} = r\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - \frac{k}{r^2} \quad -(1)$$

$$r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta + 2\dot{r}\dot{\theta} = 0 \quad -(2)$$

$$r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta = 0 \quad -(3)$$

In two dimensions these become:

$$m\ddot{r} = r\dot{\phi}^2 - \frac{k}{r^2} \quad -(4)$$

$$0 = 0 \quad -(5)$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(6)$$

In two dimensions:

$$\dot{\phi} = \frac{L}{mr^2} \quad -(7)$$

so

$$\ddot{\phi} = \frac{d\dot{\phi}}{dt} = \frac{d\dot{\phi}}{dr}\frac{dr}{dt} = \dot{r}\frac{d\dot{\phi}}{dr}$$

$$= -\frac{2L}{mr^3}\dot{r} \quad -(8)$$

So eq. (6) follows immediately, Q.E.D.

Consider now eq. (3) in three dimensions.

2)

Eq. (8) can no longer be used in three dimensions because $\dot{\phi}$ is a function of r and θ, and r and θ are not independent variables. Similarly, $\dot{\theta}$ is a function of r and ϕ, and r and ϕ are not independent variables. So in three dimensions eqs. (36) and (37) are simultaneous second order differential equations. This shows clearly that the orbit is not planar.

As the orbit approaches the planar orbit:

$$\theta \rightarrow \frac{\pi}{2} \quad -(9)$$

so

$$\sin\theta \rightarrow 1, \quad \cos\theta \rightarrow 0 \quad -(10)$$

and eqs. (2) and (3) simplify to:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(11)$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad -(12)$$

with:

$$\dot{\phi} \rightarrow \frac{L_Z}{mr^2}, \quad \dot{\theta} \rightarrow \frac{(L^2 - L_Z^2)^{1/2}}{mr^2} \quad -(13)$$

3) So in the limit (13) ϕ and θ become functions only of r, and eqs. (11) and (12) follow from eq. (13).

It is possible to evaluate the angular accelerations $\ddot{\theta}$ and $\ddot{\phi}$ as follows.

$$\ddot{\theta} = \dot{\phi}^2 \sin\theta\cos\theta - \frac{2}{r}\dot{r}\dot{\theta} \quad -(14)$$

$$\ddot{\phi} = -2\left(\frac{\dot{\theta}\dot{\phi}}{\tan\theta} + \frac{\dot{r}\dot{\phi}}{r}\right) \quad -(15)$$

where

$$r = \frac{\alpha}{1+\epsilon\cos\beta}, \quad -(16)$$

and

$$\beta = -\sin^{-1}\left(\frac{L\cos\theta}{(L^2-L_Z^2)^{1/2}}\right), \quad -(17)$$

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(18)$$

Here:

$$\dot{\phi} = \frac{L_Z}{mr^2\sin^2\theta} \quad -(19)$$

and

$$\dot{\theta} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(20)$$

Finally $\dot{r}$ is worked out from:

4)

$$H = E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r} \quad -(21)$$

So:

$$\dot{r} = \left(\frac{2}{m} \left(E - \frac{L^2}{2mr^2} + \frac{k}{r} \right) \right)^{1/2} \quad -(22)$$

Using eqs. (14) to (22) both $\ddot{\theta}$ and $\ddot{\phi}$ can be worked out in terms of θ, with input parameters E, L and L_Z, and

$$k = mMG \quad -(23).$$

The graphs of $\ddot{\theta}$ against θ and of $\ddot{\phi}$ against θ characterize the three dimensional orbit.

Planar Orbit

Here

$$\ddot{\theta} = 0 \quad -(24)$$

and

$$\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r} \quad -(25)$$

where $\dot{r}$ is given by eq. (22) and where r is given by:

5)

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} \quad (26)$$

with

$$\dot{\phi} = \frac{L}{mr^2} \quad (27)$$

so $\ddot{\phi}$ can be graphed against ϕ for a planar orbit, and this graph compared directly with the graph of $\ddot{\phi}$ against ϕ for a three dimensional orbit. The graph of $\ddot{\phi}$ against ϕ for a three dimensional orbit is found from the graph of $\ddot{\phi}$ against θ for a three dimensional orbit using:

$$\phi = -\frac{1}{2}\left(\sin^{-1}\left(\frac{(1+\cos\theta)L^2 - L_z^2}{|1+\cos\theta|\,(L^4 - L_z^2 L^2)} \right) + \sin^{-1}\left(\frac{(\cos\theta - 1)L^2 + L_z^2}{|(\cos\theta - 1)|\,(L^4 - L_z^2 L^2)} \right) \right) \quad (28)$$