271(3): Some Non-Newtonian Terms in Plane and Spherical Polar Coordinates

Plane Polar

The acceleration is:

$$\underline{a} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta\right) \quad -(1)$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

Note that for: $$\dot{\theta} = \frac{L}{mr^2} \quad -(2)$$

then $$\ddot{\theta} = \dot{r}\frac{d}{dr}\left(\frac{L}{mr^2}\right) = -2\dot{r}\frac{L}{mr^3} \quad -(3)$$

so $$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(4)$$

Therefore: $$\underline{a} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r + \underline{\omega}\times\underline{r}\right)$$

$$= \left(\ddot{r}\,\underline{e}_r + \underline{\omega}\times\left(\underline{\omega}\times\underline{r}\right)\right) \quad -(5)$$

where the centrifugal acceleration is:

$$\underline{a}_{cent} = -\underline{\omega}\times\left(\underline{\omega}\times\underline{r}\right) \quad -(6)$$

$$= \frac{L^2}{mr^3}\,\underline{e}_r = \omega^2 r\,\underline{e}_r$$

This follows from:

2)

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \underline{\omega} \times (r\dot{\theta}\,\underline{e}_\theta)$$

$$= \dot{\theta}\,\underline{k} \times r\dot{\theta}\,\underline{e}_\theta = -r\dot{\theta}^2\,\underline{e}_r \qquad -(7)$$

The Leibniz equation is therefore:

$$\underline{F} = m\ddot{r}\,\underline{e}_r = -\underline{\omega} \times (\underline{\omega} \times \underline{r}) - \frac{k}{r^2}\,\underline{e}_r$$

$$= \left(\frac{L^2}{mr^3} - \frac{k}{r^2} \right) \underline{e}_r \qquad -(8)$$

The centrifugal acceleration is non Newtonian and was first analysed correctly by Coriolis in the nineteenth century.

Spherical Polar

The total linear velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \qquad -(9)$$

$$= \underline{v}_N + \underline{\omega} \times \underline{r}$$

where the angular velocity is:

$$\underline{\omega} = \dot{\theta}\,\underline{e}_\phi - \dot{\phi}\sin\theta\,\underline{e}_\theta = \frac{\underline{L}}{mr^2} \qquad -(10)$$

where $\underline{L}$ is the total angular momentum.

). This follows from:

$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r \quad -(11)$$

$$\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta \quad -(12)$$

$$\underline{e}_r \times \underline{e}_\theta = e_\phi \quad -(13)$$

and

$$\underline{r} = r\,\underline{e}_r \quad -(14)$$

Now use:

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(15)$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(16)$$

to find that:

$$\begin{aligned}\underline{\omega} &= (-\dot{\theta}\sin\phi - \dot{\phi}\sin\theta\cos\theta\cos\phi)\,\underline{i} \\ &\quad + (\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi)\,\underline{j} \\ &\quad + \dot{\phi}\sin^2\theta\,\underline{k} \quad -(17) \\ &= \frac{1}{mr^2}(L_x\,\underline{i} + L_y\,\underline{j} + L_z\,\underline{k})\end{aligned}$$

So

$$\omega_x = \frac{L_x}{mr^2}, \quad \omega_y = \frac{L_y}{mr^2}, \quad \omega_z = \frac{L_z}{mr^2} \quad -(18)$$

<u>There are</u> <u>three components</u> of <u>angular</u> velocity <u>instead</u> of <u>one in plane polars</u>.

t) The orbital linear velocity is:

$$\underline{V} = \underline{\omega} \times \underline{r} = r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi$$

$$= r\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)\underline{i}$$

$$+ \dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\big)\underline{j}$$

$$- \dot{\theta}\sin\theta\,\underline{k} \qquad -(19)$$

In these equations:

$$\dot{\phi} = \frac{L_Z}{mr^2\sin^2\theta} \qquad -(20)$$

$$\dot{\theta} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \qquad -(21)$$

The orbital velocity is non Newtonian and is not in the X-Y plane.
