Note 271(6): Proof that Orbits are Three Dimensional in General.

Consider the angular momentum in three dimensions:

$$\underline{L} = \underset{\sim}{r} \times \underset{\sim}{p} \quad -(1)$$

Conservation of angular momentum implies that:

$$\frac{d\underline{L}}{dt} = \underline{0} \quad -(2)$$

From eqs. (1) and (2):

$$\frac{d\underline{L}}{dt} = \dot{\underset{\sim}{r}} \times \underset{\sim}{p} + \underset{\sim}{r} \times \dot{\underset{\sim}{p}} = \underline{0}. \quad -(3)$$

However,

$$\underset{\sim}{p} = m\dot{\underset{\sim}{r}} \quad -(4)$$

so

$$\frac{d\underline{L}}{dt} = m\underset{\sim}{r} \times \underline{a} = \underline{0} \quad -(5)$$

where $\underline{a}$ is the acceleration:

$$\underline{a} = \ddot{r}\,\underline{e}_r + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + 2\underset{\sim}{\omega} \times \frac{dr}{dt}\underline{e}_r. \quad -(6)$$

It was shown in the previous note that for any orbit:

$$\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + 2\underset{\sim}{\omega} \times \frac{dr}{dt}\underline{e}_r = \underline{0} \quad -(7)$$

2)

So:

$$\underset{\sim}{a} = \ddot{r}\,\underline{e}_r + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})$$

$$= \left(\ddot{r} - (r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2)\right)\underline{e}_r \quad -(8)$$

So

$$\underset{\sim}{a} \times \underset{\sim}{r} = \underline{0} \quad -(9)$$

<u>QED</u> because

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(10)$$

<u>Conservation</u> <u>of</u> <u>angular</u> <u>momentum</u> <u>implies</u> <u>eq. (7)</u> <u>and</u> <u>vice</u> <u>versa.</u>

Eq. (9) implies that the torque is zero:

$$\underline{Tq} = \frac{d\underline{L}}{dt} = \underset{\sim}{r} \times \underset{\sim}{F} = \underline{0} \quad -(11)$$

In spherical polar and Cartesian coordinates:

$$\underset{\sim}{r} = r\,\underline{e}_r = r(\sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k})$$

$$= X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(12)$$

$$\underline{p} = m\Big((\dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})\underline{i}$$

$$+ (\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\underline{j}$$

$$+ (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\underline{k}\Big)$$

$$= p_X\underline{i} + p_Y\underline{j} + p_Z\underline{k}$$

3)
$$= m\left(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi\right) \quad -(13)$$

and

$$\underline{L} = mr^2\left(\dot{\theta}\,\underline{e}_\phi - \dot{\phi}\,\underline{e}_\theta\right) = mr^2\underline{\omega}$$

$$= mr^2\Big(\left(-\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi\right)\underline{i}$$

$$+ \left(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi\right)\underline{j} \quad -(14)$$

$$+ \sin^2\theta\,\dot{\phi}\,\underline{k}$$

$$= L_X\,\underline{i} + L_Y\,\underline{j} + L_Z\,\underline{k}$$

It has been proven that in general, $\underset{\sim}{r}$ and $\underset{\sim}{p}$ do not lie in the XY plane and $\underline{L}$ is not directed along $\underline{k}$, QED. The motion of m around M does not lie in the XY plane in general.

Reduction to Planar Orbits

This occurs if and only if:

$$\theta = \pi/2 \quad -(15)$$

$$\frac{d\theta}{dt} = 0 \quad -(16)$$

Then the analysis reduces to the following.

4)

$$\underset{\sim}{r} = r(\cos\phi\, \underline{i} + \sin\phi\, \underline{j}) \quad -(17)$$

$$\underline{p} = m\left(\left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right)\underline{i} + \left(\dot{r}\sin\phi + r\dot{\phi}\cos\phi\right)\underline{j}\right) \quad -(18)$$

$$\underline{L} = -mr^2\dot{\phi}\,\underline{e}_\theta \quad -(19)$$

where:

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(20)$$

$$= -\underline{k}$$

for $\theta = \pi/2$, so

$$\underline{L} = mr^2\dot{\phi}\,\underline{k} \quad -(21)$$

Q.E.D. For planar orbits eqs. (17) and (18) show that the $\underset{\sim}{r}$ and $\underline{p}$ vectors lie in the XY plane and $\underline{L}$ is in the Z axis. Remarkably, this highly restrictive assumption (15) and (16) has been the basis for orbital theory for over four hundred years. Note carefully that none of these calculations depend on the form of the force law between m and M.