271(8): Comparison of 2-D and 3-D Orbits with the Elliptical Functions

As shown in previous work the elliptical function is:

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(1)$$

and this represents the three dimensional orbit with:

$$\beta = -\sin^{-1}\left(\frac{L \cos\theta}{(L^2 - L_Z^2)^{1/2}}\right) \quad -(2)$$

$$= \tan^{-1}\left(\frac{L}{L_Z}\tan\phi\right)$$

So:

$$\frac{L}{L_Z}\tan\phi = \tan\left(-\sin^{-1}\left(\frac{L\cos\theta}{(L^2 - L_Z^2)^{1/2}}\right)\right) \quad -(3)$$

and

$$\phi = \tan^{-1}\left(\frac{L_Z}{L}\tan\left(-\sin^{-1}\left(\frac{L\cos\theta}{(L^2 - L_Z^2)^{1/2}}\right)\right)\right) \quad -(4)$$

There is a dependence of ϕ on θ, but in a flat 2-D orbit there is no such dependence.

From eq. (2):

$$\tan\beta = \frac{L}{L_Z}\tan\phi \quad -(5)$$

2) so as:
$$L \to L_z \quad -(6)$$
then
$$\beta \to \phi \quad -(7)$$
For a 2-D planar orbit:
$$\theta = \frac{\pi}{2} = \text{constant} \quad -(8)$$
Using the Maclaurin expansion:
$$\beta + \frac{\beta^3}{3} + \frac{2}{15}\beta^5 + \dots = \frac{L}{L_z}\left(\phi + \frac{\phi^3}{3} + \frac{2}{15}\phi^5 + \dots\right). \quad -(9)$$
and from eqs. (6) and (7) the following result is obtained as the planar orbit is approached:
$$\beta = \frac{L}{L_z}\phi \quad -(10)$$
$$\frac{\beta^3}{3} = \frac{L}{L_z}\frac{\phi^3}{3} \quad -(11)$$
and so on. From eq. (10), the ellipse becomes a precessing ellipse as the planar orbit is approached:
$$r = \frac{\alpha}{1 + \epsilon\cos\left(\frac{L}{L_z}\phi\right)} \quad -(12)$$
