271(4): The Acceleration in Plane Polar Coordinates

Note that:

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(1)$$

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \quad -(2)$$

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r \quad -(3)$$

So the velocity is

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\underline{e}_r\right)$$

$$= \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \quad -(4)$$

So

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta = \dot{r}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(5)$$

The velocity is a covariant derivative:

$$\underline{v} = \frac{D\underline{r}}{dt} = \frac{dr}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(6)$$

where the angular velocity $\underline{\omega}$ is the Cartan spin connection.

In deriving eq. (5) from eq. (4) use:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(7)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(8)$$

$$\underline{k} = \underline{k} \quad -(9)$$

Therefore there is a cyclical relation:

2)

$$\underline{e}_r \times \underline{e}_\theta = \underline{k} \quad -(10)$$

$$\underline{k} \times \underline{e}_r = \underline{e}_\theta \quad -(11)$$

$$\underline{e}_\theta \times \underline{k} = \underline{e}_r \quad -(12)$$

So

$$r\dot{\theta}\,\underline{e}_\theta = \dot{\theta}\,\underline{k} \times r\,\underline{e}_r = \underline{\omega} \times \underline{r} \quad -(13)$$

The acceleration is:

$$\underline{a} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta\right)$$

$$= \ddot{r}\,\underline{e}_r + \dot{r}\,\dot{\underline{e}}_r + \dot{r}\dot{\theta}\,\underline{e}_\theta + r\ddot{\theta}\,\underline{e}_\theta + r\dot{\theta}\,\dot{\underline{e}}_\theta$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(14)$$

Now use:

$$\underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) = \underline{\omega} \times r\dot{\theta}\,\underline{e}_\theta \quad -(15)$$

$$= -r\dot{\theta}^2\,\underline{e}_r \quad -(16)$$

and

$$r\ddot{\theta}\,\underline{e}_\theta = \ddot{\theta}\,\underline{k} \times r\,\underline{e}_r = \frac{d\underline{\omega}}{dt} \times \underline{r} \quad -(17)$$

with

$$2\dot{r}\dot{\theta}\,\underline{e}_\theta = 2\underline{\omega} \times \frac{d\underline{r}}{dt} = 2\underline{\omega} \times \underline{v}$$

to find that the acceleration in plane polar coordinates is:

3)

$$\underline{a} = \ddot{r}\,\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \underline{v} \quad -(18)$$

The Newtonian acceleration is:

$$\underline{a} = \ddot{r}\,\underline{e}_r \quad -(19)$$

If eq. (18) is used with the inverse square law

$$\underline{F} = -\frac{k}{r^2}\,\underline{e}_r \quad -(20)$$

then:

$$\boxed{m\ddot{r}\,\underline{e}_r = -m\underline{\omega} \times (\underline{\omega} \times \underline{r}) - 2m\underline{\omega} \times \underline{v} - m\frac{d\underline{\omega}}{dt} \times \underline{r} - \frac{k}{r^2}\,\underline{e}_r} \quad -(21)$$

the centrifugal force is:

$$\underline{F}(\text{cent}) = -m\underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(22)$$

The Coriolis force is:

$$\underline{F}(\text{Coriolis}) = -2m\underline{\omega} \times \underline{v} \quad -(23)$$

and the third force is

$$\underline{F}_3 = -m\frac{d\underline{\omega}}{dt} \times \underline{r} \quad -(24)$$

4) These were inferred by G. G. Coriolis in 1835 from an analysis of water wheels.

For an orbit with:

$$\omega = \frac{L}{mr^2} \quad -(25)$$

it follows that:

$$2\underline{\omega} \times \underline{v} + \frac{d\underline{\omega}}{dt} \times \underline{r} = \underline{0} \quad -(26)$$

so the orbit is described by the Leibniz equation of 1689:

$$m\ddot{r}\,\underline{e}_r = -m\underline{\omega} \times (\underline{\omega} \times \underline{r}) - \frac{k}{r^2}\underline{e}_r$$

$$= \frac{L^2}{mr^3}\underline{e}_r - \frac{k}{r^2}\underline{e}_r \quad -(27)$$

All three forces (22) - (24) are real.

We now extend this analysis to spherical polar coordinates to produce hitherto unknown forces.