271(5): The Non-Newtonian Velocity and Accelerations in Spherical Polar Coordinates.

It has been shown in previous notes that the linear velocity is:

$$\underline{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \underset{\sim}{\omega} \times \underset{\sim}{r} \quad -(1)$$

where

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(2)$$

and:

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \quad -(3)$$

$$\underline{\omega} = \dot{\theta}\,\underline{e}_\phi - \dot{\phi}\sin\theta\,\underline{e}_\theta \quad -(4)$$

So:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(5)$$

The acceleration is:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r + \underline{\omega}\times\underline{r}\right)$$

$$= \ddot{r}\,\underline{e}_r + \dot{r}\,\dot{\underline{e}}_r + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times \frac{d\underline{r}}{dt}$$

$$-(6)$$

Now use:

2)

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta + \dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(7)$$

From eqs. (4) and (5):

$$\underset{\sim}{\omega} \times \frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{\omega} \times \underline{V} = \dot{r}\dot{\theta}\,\underline{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\underline{e}_\phi - \left(r\dot{\phi}^2\sin^2\theta + r\dot{\theta}^2\right)\underline{e}_r \quad -(8)$$

From eqs. (2) and (4)

$$\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) = -\left(r\dot{\phi}^2\sin^2\theta + r\dot{\theta}^2\right)\underline{e}_r \quad -(9)$$

So:

$$\underset{\sim}{\omega} \times \underline{V} = \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + \dot{r}\dot{\theta}\,\underline{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(10)$$

From eqs. (6), (7) and (10):

$$\underline{a} = \ddot{r}\,\underline{e}_r + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + 2\left(\dot{r}\dot{\theta}\,\underline{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\underline{e}_\phi\right) \quad -(11)$$

Now note that:

$$\underset{\sim}{\omega} \times \frac{dr}{dt}\,\underline{e}_r = \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{e}_\phi \\ 0 & -\dot{\phi}\sin\theta & \dot{\theta} \\ \dot{r} & 0 & 0 \end{vmatrix}$$

$$= \dot{r}\dot{\theta}\,\underline{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(12)$$

So:

$$\underline{a} = \ddot{r}\,\underline{e}_r + \underset{\sim}{\omega}\times(\underset{\sim}{\omega}\times\underset{\sim}{r}) + \dot{\underset{\sim}{\omega}}\times\underset{\sim}{r} + 2\underset{\sim}{\omega}\times\frac{dr}{dt}\underline{e}_r \quad -(13)$$

This is exactly the same result as in plane polar coordinates, but the internal structure is much richer.

Main Results

1) The angular velocity or Cartan spin connection is:

$$\underset{\sim}{\omega} = \dot{\theta}\,\underline{e}_\phi - \dot{\phi}\sin\theta\,\underline{e}_\theta$$

$$= \frac{\underline{L}}{mr^2} \quad -(14)$$

where $\underline{L}$ is the total angular momentum:

$$\underline{L} = L_x\,\underline{i} + L_y\,\underline{j} + L_z\,\underline{k} \quad -(15)$$

It has three Cartesian components in general.

2) The orbital velocity is:

4)

$$\underline{v}_{\text{orbital}} = \underset{\sim}{\omega} \times \underset{\sim}{r}$$

$$= \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(16)$$

and is three dimensional in general. To convert to Cartesian coordinates use:

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \quad -(17)$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(18)$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(19)$$

Here:

$$\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta \quad -(20)$$

$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r \quad -(21)$$

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi \quad -(22)$$

It is clear that $\underline{v}_{\text{orbital}}$ has three Cartesian components and so is three dimensional.

3) The Centrifugal Acceleration

This is:

$$\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) = -\left(r\dot{\phi}^2\sin^2\theta + r\dot{\theta}^2\right)\underline{e}_r \quad -(23)$$

and this has three Cartesian components so is three dimensional.

4) The Coriolis Acceleration

This is:

$$\underline{a}\,(\text{Coriolis}) = 2\underset{\sim}{\omega} \times \frac{dr}{dt}\,\underline{e}_r \qquad -(24)$$

$$= 2\left(\dot{r}\dot{\theta}\,\underline{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\underline{e}_\phi\right)$$

and this is also three dimensional in general.

5) The Acceleration due to $\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}$

This is evaluated using:

$$\dot{\underset{\sim}{\omega}} = \frac{d}{dt}\left(\dot{\theta}\,\underline{e}_\phi - \dot{\phi}\sin\theta\,\underline{e}_\theta\right) \qquad -(25)$$

$$= \ddot{\theta}\,\underline{e}_\phi + \dot{\theta}\,\dot{\underline{e}}_\phi - \ddot{\phi}\sin\theta\,\underline{e}_\theta - \dot{\phi}\dot{\theta}\cos\theta\,\underline{e}_\theta - \dot{\phi}\sin\theta\,\dot{\underline{e}}_\theta$$

Using (VAPS 21-24):

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta + \sin\theta\,\dot{\phi}\,\underline{e}_\phi \qquad -(26)$$

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r + \cos\theta\,\dot{\phi}\,\underline{e}_\phi \qquad -(27)$$

$$\dot{\underline{e}}_\phi = -\sin\theta\,\dot{\phi}\,\underline{e}_r - \cos\theta\,\dot{\phi}\,\underline{e}_\theta \qquad -(28)$$

Therefore:

6)

$$\dot{\underset{\sim}{\omega}} = \underline{e}_\phi \left(\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta \right) - \underline{e}_\theta \left(2 \dot{\phi} \dot{\theta} \cos\theta + \ddot{\phi} \sin\theta \right) \quad -(29)$$

and

$$\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} = r \left(\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta \right) \underline{e}_\theta + r \left(\ddot{\phi} \sin\theta + 2 \dot{\theta} \dot{\phi} \cos\theta \right) \underline{e}_\phi \quad -(30)$$

<u>This is</u> <u>again</u> three <u>dimensional</u> <u>in general</u>.

The Leibniz orbital equation is:

$$m \ddot{r} \underline{e}_r = -m \left(\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + 2 \underline{\omega} \times \frac{dr}{dt} \underline{e}_r + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} \right) - \frac{k}{r^2} \underline{e}_r \quad -(31)$$

which splits into two equations:

$$m \ddot{r} = r \left(\dot{\theta}^2 + r \dot{\phi}^2 \sin^2\theta \right) - \frac{k}{r^2} \quad -(32)$$

and:

$$2 \underset{\sim}{\omega} \times \frac{dr}{dt} \underline{e}_r + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} = 0 \quad -(33)$$

Eq. (32) is true for each component of $\underline{e}_r$,

7) So $m\ddot{r}\underline{e}_r$ has three Cartesian components:

$$m\ddot{r}\,\underline{e}_r = m\ddot{r}\left(\sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}\right) \quad -(34)$$

so the orbit is three dimensional.

The orbit is constrained by eq. (33), which gives:

$$2\left(\dot{r}\dot{\theta}\,\underline{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\underline{e}_\phi\right) + r\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right)\underline{e}_\theta + r\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right)\underline{e}_\phi = \underline{0} \quad -(35)$$

i.e.

$$r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta + 2\dot{r}\dot{\theta} = 0 \quad -(36)$$

and

$$r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta = 0 \quad -(37)$$

From UFT 270:

$$\dot{\phi} = \frac{L_Z}{mr^2\sin^2\theta} \quad -(38)$$

$$\dot{\theta} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(39)$$