270(7): Check on the Validity of the Lagrangian Method

The basic method is to find the maximum or minimum of the integral:

$$J = \int_{x_1}^{x_2} f(y(x), y'(x); x)\, dx \quad -(1)$$

The solution is Euler's equation:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'} \quad -(2)$$

If there are several dependent variables then:

$$f = f(y_1(x), y_1'(x), y_2(x), y_2'(x), \ldots; x) \quad -(3)$$

$$= f(y_i(x), \dot{y}_i(x); x), \quad i = 1, 2, 3, \ldots$$

and

$$\frac{\partial f}{\partial y_i} = \frac{d}{dx}\frac{\partial f}{\partial y_i'} \quad -(4)$$

Hamilton's principle is used with the calculus of variations:

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \quad -(5)$$

$$= \delta \int_{t_1}^{t_2} \mathcal{L}(x_i, \dot{x}_i; t)\, dt$$

to give the Euler Lagrange equations:

2)

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \quad ; \; i = 1, 2, 3, \ldots \qquad -(6)$$

For the spherical polar coordinates:

$$\mathcal{L} = T - V \qquad -(7)$$

where:

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\right)$$

$$= \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\beta}^2\right) \qquad -(8)$$

is the kinetic energy.

For an inverse square law:

$$V = -\frac{k}{r} \qquad -(9)$$

The potential (9) produces planar orbits in spherical polar coordinates. The reason for this was given in eq. (17) of note 270(3):

$$\beta = \frac{L}{L_\theta}\theta = \frac{L}{L_\phi}\phi \sin\theta \qquad -(10)$$

and

$$\dot{\beta} = \frac{L}{L_\theta}\dot{\theta} \qquad -(11)$$

$$= \frac{L}{L_\phi}\left(\dot{\phi}\sin\theta + \phi\dot{\theta}\cos\theta\right)$$

so

$$\theta = \frac{\pi}{2} \qquad -(12)$$

3) Here L, L_θ and L_ϕ are all constants of motion, where
$$L^2 = L_\theta^2 + L_\phi^2 \quad -(13)$$

The Lagrangian can be expressed as:
$$\mathcal{L} = \mathcal{L}\left(r(t), \dot{r}(t), \beta(t), \dot{\beta}(t) ; t \right) - (14)$$

or as:
$$\mathcal{L} = \mathcal{L}\left(r(t), \dot{r}(t), \theta(t), \dot{\theta}(t), \phi(t), \dot{\phi}(t) ; t \right) - (15)$$

From eq. (14):
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(16)$$

and
$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad -(17)$$

Eqs. (16) and (17) give:
$$m\left(\ddot{r} - r\dot{\beta}^2\right) = -\frac{k}{r^2} \quad -(18)$$

and
$$\dot{\beta} = \frac{L}{mr^2} \quad -(19)$$

The solution of eqs. (18) and (19) is the conical section:
$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(20)$$

4)

where:

$$\alpha = \frac{L^2}{mk} , \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(21)$$

with:

$$H = E = T + V \quad -(22)$$

is the hamiltonian. If the conical section is an ellipse, the total energy can be measured from the semi-major axis a:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2E} . \quad -(23)$$

The semi minor axis is defined by:

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2mE)^{1/2}} \quad -(24)$$

So both L and E can be measured for an elliptic orbit.

The lagrangian (15) gives eq. (16) and two more equations:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(25)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(26)$$

Eq. (25) gives:

5)

$$L_\theta = mr^2 \frac{d\theta}{dt} \quad -(27)$$

and:

$$\sin\theta \frac{d^2\phi}{dt^2} + 2\cos\theta \frac{d\theta}{dt}\frac{d\phi}{dt} = 0 \quad -(28)$$

The total angular momentum is given by:

$$L^2 = L_\theta^2 + L_\phi^2 \quad -(29)$$

Fm Eqs. (19) and (27):

$$m^2 r^4 \dot{\beta}^2 = m^2 r^4 \dot{\theta}^2 + L_\phi^2 \quad -(30)$$

Fm eq. (8) it follows that:

$$L_\phi = mr^2 \sin\theta \frac{d\phi}{dt} \quad -(31)$$

Eq. (10) then follows fm eqs. (19), (27) and (31) provided that L, L_θ and L_ϕ are all constants of motion.

This is the fundamental reason for planar orbits.

If this condition is broken the orbit becomes three dimensional. The condition can be broken by an external torque in θ.

). The torque is defined by

$$Tq = -\frac{\partial V(\theta)}{\partial \theta} \quad -(32)$$

where $V(\theta)$ is a potential included in the lagrangian as follows:

$$\mathcal{L} = T - V(r) - V(\theta) \quad -(33)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\beta}^2) + \frac{k}{r} - V(\theta) \quad -(33a)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)) + \frac{k}{r} - V(\theta). \quad -(33b)$$

With this lagrangian, eq. (16) gives eq (18) again. Eq. (25) gives:

$$Tq = -\frac{\partial V(\theta)}{\partial \theta} = mr^2 \frac{d^2\theta}{dt^2} + 2mr \frac{dr}{dt}\frac{d\theta}{dt} \quad -(34)$$

so the external torque means that L_θ is no longer a constant of motion:

$$\boxed{\frac{dL_\theta}{dt} \neq 0} \quad -(35)$$

7) This is the condition for three dimensional orbits.

It remains to consider eq. (17) with the lagrangian (33a). From these two equations:

$$\frac{\partial \mathcal{L}}{\partial \beta} = -\frac{\partial V(\theta)}{\partial \beta} \quad -(36)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\beta}} = \frac{d}{dt}\left(mr^2\dot{\beta}\right) \quad -(37)$$

So

$$\boxed{\frac{d(mr^2\dot{\beta})}{dt} = -\frac{\partial V(\theta)}{\partial \beta}} \quad -(38)$$

In general: $-(39)$

$$\frac{d}{dt}\left(mr^2\dot{\beta}\right) = mr^2\frac{d^2\beta}{dt^2} + 2m\frac{dr}{dt}\frac{d\beta}{dt}$$

Therefore a potential of the type $V(\theta)$ means that L is no longer conserved.

So the system of equations is:

$$m\left(\ddot{r} - r\dot{\beta}^2\right) = -\frac{k}{r^2} \quad -(39)$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -\frac{\partial V(\theta)}{\partial \theta} \quad -(40)$$

and
$$\frac{d}{dt}\left(mr^2\dot{\beta}\right) = -\frac{\partial V(\theta)}{\partial \beta} \quad -(41)$$

However the equation (41) has been obtained with the lagrangian: $\mathcal{L} = (r, \dot{r}, \beta, \dot{\beta}\,; t)$ $-(42)$

in which θ does not appear as a generalized lagrangian coordinate. The generalized coordinates are r and β. Now use:

$$\frac{\partial V(\theta)}{\partial \beta} = \frac{\partial V(\theta)}{\partial \theta}\frac{d\theta}{d\beta} \quad -(43)$$

where
$$T_{q_{V1}} = -\frac{\partial V(\theta)}{\partial \beta} \quad -(44)$$

and
$$T_{q_{V2}} = -\frac{\partial V(\theta)}{\partial \theta} \quad -(45)$$

so
$$\boxed{\frac{d\theta}{d\beta} = \frac{T_{q_{V1}}}{T_{q_{V2}}}} \quad -(46)$$

The additional potential $V(\theta)$ results in two torques, the ratio of which gives $\frac{d\theta}{d\beta}$. The orbit is no longer planar. If $T_{q_{V1}}$ is very small, the previous results for a precessing planar orbit are obtained

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