Note 270(12): Summary

The Hamiltonian is:

$$H = E = \frac{1}{2} m v^2 - \frac{k}{r} \quad -(1)$$

In spherical polar coordinates:

$$v^2 = \dot{r}^2 + r^2 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) \quad -(2)$$

Define:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \quad -(3)$$

then

$$v^2 = \dot{r}^2 + r^2 \dot{\beta}^2 \quad -(4)$$

and

$$E = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\beta}^2 \right) - \frac{k}{r} \quad -(5)$$

whose solution is <u>the main orbit</u>.

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(6)$$

The lagrangian is:

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{k}{r} \quad -(7)$$

and the Euler Lagrange equations give:

$$\frac{d\beta}{dt} = \frac{L}{m r^2} \quad -(8)$$

and

2)

$$\frac{d\phi}{dt} = \frac{L_Z}{mr^2\sin^2\theta} \quad -(9)$$

Fundamental geometry also gives:

$$L_Z = mr^2\sin^2\theta \frac{d\phi}{dt} \quad -(10)$$

and

$$L^2 = m^2r^4\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$$

$$= m^2r^4\dot{\theta}^2 + \frac{L_Z^2}{\sin^2\theta} \quad -(11)$$

so

$$\frac{d\theta}{dt} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(12)$$

Eqs. (8), (9) and (12) are the three angular velocities. If:

$$\theta = \frac{\pi}{2} \quad -(13)$$

then:

$$\frac{d\beta}{dt} = \frac{d\phi}{dt} = \frac{L_Z}{mr^2} \quad -(14)$$

and

$$\frac{d\theta}{dt} = 0 \quad -(15)$$

so the three angular velocities reduce to one angular velocity.

), In order to find the sub orbits $r = f(\theta)$, $r = f(\phi)$ and $\phi = f(\theta)$ eq. (3) must be integrated. This can be done using:

$$\frac{d\beta}{d\phi} = \frac{d\beta}{dt}\frac{dt}{d\phi} = \frac{L}{L_z}\sin^2\theta \quad -(9)$$

and

$$\frac{d\beta}{d\theta} = \frac{L}{\left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2}} \quad -(10)$$

so

$$\frac{d\phi}{d\theta} = \frac{L_z}{\left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)\sin^2\theta} \quad -(11)$$

Therefore

$$\beta = \int \frac{L\,d\theta}{\left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2}} \quad -(12)$$

and

$$\phi = \int \frac{L_z\,d\theta}{\left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2}\sin^2\theta} \quad -(13)$$

Eqs. (12) and (13) can be evaluated by computer algebra directly

4) The result for eq. (12) is:

$$\beta = -\sin^{-1}\left(\frac{L\cos\theta}{(L^2 - L_z^2)^{1/2}}\right) \quad -(14)$$

for

$$\theta \neq \frac{\pi}{2} \quad -(15)$$

This result can be double checked by evaluating eq. (12) without taking $\sin^2\theta$ out of the square root sign. This would remove any problem with the range of $\theta$.

Eq. (14) implies:

$$\cos\theta = -\frac{(L^2 - L_z^2)^{1/2}}{L}\sin\beta \quad -(16)$$

and

$$\sin^2\theta = 1 + \frac{L_z^2}{L^2}\sin^2\beta \quad -(17)$$

From eqs. (9) and (17):

$$\phi = \int \frac{L_z}{L}\left(1 + \frac{L_z^2}{L^2}\sin^2\beta\right)^{-1} d\beta$$

$$= \int \frac{L_z \, d\beta}{L\left(1 + \frac{L_z^2}{L^2}\sin^2\beta\right)} \quad -(18)$$

5) Therefore $\phi$ can be found as a function of $\beta$. Hand calculation gives the result:

$$\phi = \frac{L}{L_Z a} \tan^{-1}(a \tan \beta) \quad -(19)$$

where

$$a = \left(1 + \left(\frac{L_Z}{L}\right)^2\right)^{1/2} \quad -(20)$$

Eq. (19) can be checked by computer algebra for any human error. From eq. (19):

$$\tan^{-1}(a \tan \beta) = a \frac{L_Z}{L} \phi \quad -(21)$$

So

$$a \tan \beta = \tan\left(a \frac{L_Z}{L} \phi\right) \quad -(22)$$

and

$$\beta = \tan^{-1}\left(\frac{1}{a} \tan\left(a \frac{L_Z}{L} \phi\right)\right) \quad -(23)$$