270(11): Functional Relation between β and ϕ.

From computer algebra, both β and ϕ can be expressed in terms of θ:

$$\beta = - \sin^{-1}\left(\frac{L\cos\theta}{(L^2-L_Z^2)^{1/2}}\right) \quad -(1)$$

$$\phi = -\frac{1}{2}\left(\sin^{-1}\left(\frac{(1+\cos\theta)L^2-L_Z^2}{|1+\cos\theta|(L^4-L_Z^2L^2)}\right)\right. \quad -(2)$$

$$\left. + \sin^{-1}\left(\frac{(\cos\theta-1)L^2+L_Z^2}{|(\cos\theta-1)|(L^4-L_Z^2L^2)}\right)\right)$$

So from eq. (1):

$$\cos\theta = -\frac{(L^2-L_Z^2)^{1/2}}{L}\sin\beta \quad -(3)$$

From eqns. (2) and (3) ϕ can be expressed as a function of β. It may be possible to invert

$$\phi = f(\beta) \quad -(4)$$

to find

$$\beta = f(\phi) \quad -(5)$$

2)

From Eq. (29) of Note 270(10):

$$\beta = \frac{L}{L_z} \int \sin^2\theta \, d\phi \quad -(6)$$

where

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \frac{(L^2 - L_z^2)}{L^2} \sin^2\beta$$

$$= 1 + \frac{L_z^2}{L^2} \sin^2\beta \quad -(7)$$

So

$$\beta = \frac{L}{L_z} \int \left(1 + \frac{L_z^2}{L^2} \sin^2\beta\right) d\phi \quad -(8)$$

and

$$\frac{d\beta}{d\phi} = \frac{L_z}{L} \left(1 + \frac{L_z^2}{L^2} \sin^2\beta \right) \quad -(9)$$

so

$$\phi = \frac{L}{L_z} \int \frac{d\beta}{1 + \frac{L_z^2}{L^2} \sin^2\beta} \quad -(10)$$

$$= \frac{L}{L_z} \frac{\tan^{-1}\left(\left(1 + (L_z/L)^2\right)^{1/2} \tan\beta\right)}{\left(1 + \left(\frac{L_z}{L}\right)^2\right)^{1/2}} \quad -(11)$$

3) Therefore, defining:

$$a = \frac{L_z}{L} \quad -(12)$$

$$\phi = \frac{1}{a(1+a^2)^{1/2}} \tan^{-1}\left((1+a^2)^{1/2} \tan\beta\right) \quad -(13)$$

and

$$(1+a^2)^{1/2} \tan\beta = \tan\left(a(1+a^2)^{1/2} \phi\right) \quad -(14)$$

$$\beta = \tan^{-1}\left[\frac{1}{(1+a^2)^{1/2}} \tan\left(a(1+a^2)^{1/2} \phi\right)\right] \quad -(15)$$

and $$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(16)$$