270(2): Interpretation for Constant θ

In general there are three equations:

$$\frac{d\beta}{dt} = \frac{L}{mr^2}, \quad -(1)$$

$$\frac{d\theta}{dt} = \frac{L_\theta}{mr^2} \quad -(2)$$

and

$$\frac{d\phi}{dt} = \frac{L_\phi}{mr^2 \sin\theta}. \quad -(3)$$

So when

$$\frac{d\theta}{dt} \neq 0 \quad -(4)$$

it follows that:

$$\beta = \frac{L}{L_\theta}\theta = \frac{L}{L_\phi}\phi\sin\theta \quad -(5)$$

and

$$\theta = \frac{L_\theta}{L_\phi}\phi\sin\theta \quad -(6)$$

However, when

$$\frac{d\theta}{dt} = 0 \quad -(7)$$

then there are only two equations (1) and (2) because the Euler Lagrange equation in θ and $\dot{\theta}$ cannot be defined when eq. (7) is true. In this case:

$$\beta = \left(\frac{L}{L_\phi}\sin\theta\right)\phi = x\phi \quad -(8)$$

) and the precessing planar ellipse is:

$$r = \frac{\alpha}{1 + \epsilon \cos x\phi} \quad -(9)$$

Q.E.D.

In the ellipse (9):

$$L \sim L_\phi \quad -(10)$$

and so

$$x = \sin\theta \quad -(11)$$

Q.E.D.

In the general three dimensional orbit, θ and ϕ are related by Eq. (6), i.e.

$$\frac{\sin\theta}{\theta} = \left(\frac{L_\phi}{L_\theta}\right)\frac{1}{\phi} \quad -(12)$$

But for constant θ, eq. (6) no longer applies.

Using the Maclaurin series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \quad -(13)$$

So

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^3}{5!} - \dots \quad (14)$$

$$= \left(\frac{L_\phi}{L_\theta}\right)\frac{1}{\phi} \quad -(15)$$

3) The motion of the θ ellipse and the ϕ ellipse are always linked by eq. (6) for $d\theta/dt \neq 0$, so it is sufficient to graph:

$$r = \frac{\alpha}{1 + \epsilon \cos\left(\left(\frac{L}{L_\phi}\sin\theta\right)\phi\right)} \quad -(16)$$

in a polar plot. This is a three dimensional plot of r against θ and ϕ. It transforms into a two dimensional plot when:

$$x = \sin\theta = 1 + \frac{3MG}{c^2\alpha} \quad -(17)$$

and

$$L \sim L_\phi \quad -(18)$$

There appears to be no reason why a 3-D orbit should become a 2-D orbit, but the geometry shows that all 2-D orbits must precess. The inverse square law of attraction in 3-D always transform to a precessing 2-D orbit.