Note 270(13): Final Relation between β and ϕ.

Eq. (14) of note 270(12) implies that:

$$\cos\theta = -\frac{(L^2-L_Z^2)^{1/2}}{L}\sin\beta \quad -(1)$$

and

$$\sin^2\theta = 1-\frac{(L^2-L_Z^2)}{L^2}\sin^2\beta \quad -(2)$$

so:

$$\frac{d\beta}{d\phi} = \frac{L}{L_Z}\sin^2\theta$$
$$= \frac{L}{L_Z}\left(1-\frac{1}{L^2}\left(L^2-L_Z^2\right)\sin^2\beta\right) \quad -(3)$$

Therefore:

$$\phi = \int d\phi = \frac{L_Z}{L}\int\left(1-\frac{1}{L^2}\left(L^2-L_Z^2\right)\sin^2\beta\right)^{-1}d\beta$$
$$= \tan^{-1}\left(\frac{L_Z}{L}\tan\beta\right) \quad -(4)$$

using Maxima code by Dr. Horst Eckardt.

So:

$$\boxed{\beta = \tan^{-1}\left(\frac{L}{L_Z}\tan\phi\right)} \quad -(5)$$

The theory of planar orbits assume that

$$\beta = \phi \quad -(6)$$

and

$$L = L_z \quad -(7)$$

Final Results

$$\beta = \tan^{-1}\left(\frac{L}{L_z}\tan\phi\right), \quad -(8)$$

$$\beta = -\sin^{-1}\left(\frac{L\cos\theta}{\left(L^2 - L_z^2\right)^{1/2}}\right), \quad -(9)$$

$$\theta \neq \frac{\pi}{2}.$$

In the case of planar orbits:

$$\frac{d\theta}{d\beta} = 0 \quad -(10)$$

because

$$\theta = \pi/2 \quad -(11)$$

So in planar orbits β does not depend on θ, and

$$\beta = \phi \quad -(12)$$

Planar orbits are simplistic.