270(9): Equations for Animation

The conserved angular momenta are:

$$L = mr^2 \dot{\beta} \quad -(1)$$

and

$$L_\phi = mr^2 \dot{\phi} \sin^2\theta \quad -(2)$$

The Hamiltonian:

$$H = E = T + V = \frac{1}{2} m (\dot{r}^2 + \dot{\beta}^2 r^2) - \frac{k}{r} \quad -(3)$$

produces the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(4)$$

From eqs. (1) and (2):

$$\beta = \frac{L}{L_\phi} \sin^2\theta \, \phi \quad -(5)$$

so the orbit is:

$$r = \frac{\alpha}{1 + \epsilon \cos\left(\frac{L}{L_\phi} \sin^2\theta \, \phi\right)} \quad -(6)$$

$$= \frac{\alpha}{1 + \epsilon \cos x\phi}$$

where

$$x = \frac{L}{L_\phi} \sin^2\theta. \quad -(7)$$

From the Binet equation the Hamiltonian for eq. (6) is:

$$H = \frac{p^2}{2m} - x^2 \frac{k}{r} + (x^2 - 1) \frac{L^2}{2mr^2} \quad -(8)$$