270(10): Conserved Angular Momenta and Relation Between p, θ and ϕ

From first principles: $-(1)$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$
$$= m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta + \dot{\phi}^2 \sin^4\theta)$$

is a conserved quantity:

$$\frac{dL^2}{dt} = 0 \quad -(2)$$

From first principles:

$$L_z^2 = m^2 r^4 \dot{\phi}^2 \sin^4\theta \quad -(3)$$

and

$$L_x^2 + L_y^2 = m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta) \quad -(4)$$

These are also conserved:

$$\frac{dL_z^2}{dt} = \frac{d}{dt}(L_x^2 + L_y^2) = 0 \quad -(5)$$

Eq (1) can be written as:

$$L^2 = m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta + \dot{\phi}^2 \sin^4\theta)$$
$$= m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta (\cos^2\theta + \sin^2\theta))$$

2) $= m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$ — (6)

So ~~of~~ total angular momentum is:

$$L^2 = m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \quad -(7)$$

The lagrangian is therefore:

$$\mathcal{L} = T - V = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right) + \frac{k}{r}$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{k}{r} \quad -(8)$$

The Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(9)$$

$-(10)$

gives: $\frac{d}{dt}\left(mr^2\dot{\phi}\sin^2\theta\right) = \frac{dL_z}{dt} = 0$

Eqs (3) and (10) give ~~to~~ same result.

The Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(11)$$

gives:

3)

$$\frac{d}{dt}\left(mr^2 \dot{\theta}\right) = mr^2 \sin\theta \cos\phi \, \dot{\phi}^2 \quad -(12)$$

so the angular momentum $mr^2\dot{\theta}$ is not conserved

From first principles:

$$\underline{L} = L_x \underline{i} + L_y \underline{j} + L_z \underline{k} \quad -(13)$$

where:

$$L_x = -mr^2 \left(\dot{\theta} \sin\phi + \dot{\phi} \sin\theta \cos\theta \cos\phi\right) \quad -(14)$$

$$L_y = mr^2 \left(\dot{\theta} \cos\phi - \dot{\phi} \sin\theta \cos\theta \sin\phi\right) \quad -(15)$$

$$L_z = mr^2 \dot{\phi} \sin^2\theta \quad -(16)$$

These are conserved because:

$$\frac{d\underline{L}}{dt} = \frac{dL_x}{dt} \underline{i} + \frac{dL_y}{dt} \underline{j} + \frac{dL_z}{dt} \underline{k}$$

$$= 0 \quad -(17)$$

so

$$\frac{dL_x}{dt} = \frac{dL_y}{dt} = \frac{dL_z}{dt} = 0 \quad -(18)$$

Therefore as in quantum mechanics L^2 and L_z are conserved:

4)

$$L^2 = m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \quad -(19)$$

$$L_Z = m r^2 \dot{\phi}^2 \sin^2\theta \quad -(20)$$

In quantum mechanics:

$$L^2 \psi = \hbar^2 l(l+1) \psi \quad -(21)$$

$$L_Z \psi = m_l \hbar \psi \quad -(22)$$

From eqs. (19) and (20):

$$L^2 = m^2 r^4 \dot{\theta}^2 + \frac{L_Z^2}{\sin^2\theta} \quad -(23)$$

so

$$\dot{\theta}^2 = \left(L^2 - \frac{L_Z^2}{\sin^2\theta} \right) / m^2 r^4 \quad -(24)$$

and $\dot{\theta}^2$ can be expressed in terms of two conserved quantities, L^2 and L_Z^2.

There also exist:

$$\dot{\beta} = \frac{L}{mr^2} \quad -(25)$$

and

$$\dot{\phi} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(26)$$

where

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(27)$$

5) As in previous work the 3-D orbit is:

$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(28)$$

From eqs. (25) and (26):

$$\frac{d\beta}{d\phi} = \frac{L}{L_Z} \sin^2\theta \quad -(29)$$

From eqs. (24) and (26):

$$\frac{d\phi}{d\theta} = \frac{L_Z}{mr^2 \sin^2\theta} \, mr^2 \left(L^2 - \frac{L_Z^2}{\sin^2\theta} \right)^{-1/2} \quad -(30)$$

Therefore there exists a relation between ϕ and θ:

$$\phi = L_Z \int \frac{d\theta}{\left(L^2 \sin^2\theta - L_Z^2 \right)^{1/2} \sin\theta} \quad -(31)$$

There also exists a relation between β and θ

$$\frac{d\beta}{d\theta} = \frac{d\beta}{d\phi} \frac{d\phi}{d\theta} \quad -(32)$$

6) Therefore:

$$\beta = L \int \frac{\sin\theta \; d\theta}{(L^2 \sin^2\theta - L_Z^2)^{1/2}} \quad -(33)$$

The three dimensional orbit is therefore:

$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(34)$$

where β is given by Eq. (33).

The transition to the planar orbit is defined by:

$$\theta \rightarrow \frac{\pi}{2} \quad -(35)$$

so

$$\beta \rightarrow \left(\frac{L}{(L^2 - L_Z^2)^{1/2}} \right) \theta \quad -(36)$$

In order to find the relation between β and ϕ, β can be expressed in terms of θ as in Eq. (33) and θ can be expressed in terms of ϕ using the inverse of Eq. (31).