Note 270(5): More Details of Note 270(4)

The orbit of the hyperbolic spiral is considered to be:

$$\frac{1}{r} = \frac{\beta}{r_0} \quad -(1)$$

From the Euler Lagrange equations:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(2)$$

and

$$\frac{d\phi}{dt} = \frac{L_\phi}{mr^2 \sin\theta} \quad -(3)$$

so from eq. (3):

$$\frac{1}{mr^2} = \frac{\sin\theta}{L_\phi}\frac{d\phi}{dt} \quad -(4)$$

In Eq. (1):

$$\frac{d\beta}{dt} = \frac{L}{L_\phi}\sin\theta\frac{d\phi}{dt} \quad -(5)$$

This is eq. (10) of note 270(4).

From eqs. (1) and (2):

$$\frac{d\beta}{dt} = \frac{L}{mr_0^2}\beta^2 \quad -(6)$$

so from eqs (5) and (6):

$$\frac{\beta^2}{mr_0^2} = \frac{\sin\theta}{L_\phi}\frac{d\phi}{dt} \quad -(7)$$

So:

$$\frac{d\phi}{dt} = \frac{L_\phi}{m r_0^2 \sin\theta} \beta^2 \quad -(8)$$

Now use:

$$\beta = \frac{L}{L_\phi} \phi \sin\theta \quad -(9)$$

so

$$\frac{d\phi}{dt} = \frac{L_\phi}{m r_0^2 \sin\theta} \cdot \frac{L^2}{L_\phi^2} \phi^2 \sin^2\theta$$

$$= \left(\frac{L^2}{m r_0^2 L_\phi} \right) \sin\theta \ \phi^2 \quad -(10)$$

therefore:

$$\frac{1}{\sin\theta} \int \frac{d\phi}{\phi^2} = A \int dt \quad -(11)$$

i.e.

$$\boxed{\phi = -\frac{1}{A t \sin\theta}} \quad -(12)$$

where

$$A = \frac{L^2}{m r_0^2 L_\phi} \quad -(13)$$

If there is a constant of integration C:

$$\phi = -\frac{1}{A t \sin\theta} + C \quad -(14)$$