270(3): Transition to Planar Orbit

The basic equations of the 3-D orbit are:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) + V(r) \quad - (1)$$

where $V(r)$ is any central potential of attraction. For the inverse square force:

$$V(r) = -\frac{k}{r} \quad - (2)$$

Here:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad - (3)$$

The force of attraction is:

$$F(r) = -\frac{\partial V(r)}{\partial r} \quad - (4)$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad - (5)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad - (6)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad - (7)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad - (8)$$

2) Eq. (5) gives:

$$m\ddot{r} = mr\dot{\beta}^2 - \frac{\partial V(r)}{\partial r} \quad -(9)$$

which is the Leibniz equation.

Eqs. (6) and (7) give:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(10)$$

and

$$\frac{d\theta}{dt} = \frac{L_\theta}{mr^2} \quad -(11)$$

which imply:

$$\frac{d\phi}{dt} = \frac{L_\phi}{mr^2 \sin\theta} \quad -(12)$$

where

$$L^2 = L_\theta^2 + L_\phi^2 \quad -(13)$$

The Euler Lagrange equation (8) gives:

$$\ddot{\phi} \sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta = 0 \quad -(14)$$

From eqs. (10) and (11):

$$\frac{d\beta}{d\theta} = \frac{L}{L_\theta} \quad -(15)$$

From eqs (10) and (12)

$$\frac{d\beta}{d\phi} = \frac{L}{L_\phi}\sin\theta \quad -(16)$$

3) If constants of integration are assumed to be zero then:

$$\beta = \frac{L}{L_\theta}\theta = \frac{L}{L_\phi}\phi\sin\theta \quad -(17)$$

as in UFT 269. However, from eq. (17):

$$\dot{\beta} = \frac{L}{L_\theta}\dot{\theta} = \frac{L}{L_\phi}\left(\dot{\phi}\sin\theta + \phi\dot{\theta}\cos\theta\right) \quad -(18)$$

so

$$\dot{\theta} = \frac{L_\theta}{L_\phi}\left(\dot{\phi}\sin\theta + \phi\dot{\theta}\cos\theta\right) \quad -(19)$$

From (11), (12) and (19)

$$L_\theta = \frac{L_\theta}{L_\phi}\left(L_\phi + \phi\cos\theta\, L_\theta\right) \quad -(20)$$

so

$$\phi\cos\theta = 0 \quad -(21)$$

and

$$\theta = \pi/2 \quad -(22)$$

So the solution (17) leads back to a planar orbit. Eq. (17) is therefore the reason for planar orbits.

In order to obtain non planar orbits

4) constants of integration must be used, in the simplest case:

$$\beta = \frac{L}{L_\theta}\,\theta + A(t) \quad -(23)$$

and

$$\beta = \frac{L}{L_\phi}\,\phi \sin\theta \quad -(24)$$

where $A(t)$ is time dependent but not θ dependent. Then eq. (18) is no longer true and θ is no longer fixed at $\pi/2$.

Three Dimensional Orbits in General

In general choose the solution (24), so for an inverse square law:

$$r = \frac{\alpha}{1 + \epsilon \cos\left(\frac{L}{L_\phi} \sin\theta\, \phi\right)} \quad -(25)$$

and previous results are all valid.

For the general orbit such as those in 3-D galaxies the 3-D Binet equation is:

$$F(r) = -\frac{\partial V(r)}{\partial r} = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(26)$$

) in which:

$$\beta = \frac{L}{L_\phi} \phi \sin\theta \quad -(27)$$

and

$$m\ddot{r} = -\frac{L^2}{mr^2}\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) \quad -(28)$$

For the inverse square law of attraction:

$$F(r) = -\frac{k}{r^2} \quad -(29)$$

the orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(30)$$

which is the <u>precessing</u> ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\left(\left(\frac{L}{L_\phi}\sin\theta\right)\phi\right)} \quad -(31)$$

Under the condition (17), i.e.:

$$\theta = \frac{L_\theta}{L_\phi}\phi \sin\theta \quad -(32)$$

eq. (31) reduces to the planar ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(33)$$

for all L_θ and L_ϕ.

6) For other types of force of attraction, the 3-D Binet equation (26) produces different three dimensional orbits. These orbits are observed in three dimensional galaxies. They depend on the function:

$$f(r, \phi, \theta) = \frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) \quad -(27)$$

where

$$\beta = \frac{L}{L_\phi} \phi \sin\theta \quad -(28)$$

For a given choice of orbit the force law can be determined. Since β is a function of both ϕ and θ, this is a problem in the function of two variables, so the chain rule of differential calculus is needed.
