270(6): Three Dimensional Orbits from a Background Stochastic Cosmic Force

Consider a system in which the total angular momentum is conserved, and defined by:

$$L^2 = L_\theta^2 + L_\phi^2. \quad -(1)$$

Assume that:

$$\frac{dL_\phi}{dt} = 0 \quad -(2)$$

but

$$\frac{dL_\theta}{dt} \neq 0 \quad -(3)$$

The Lagrangian for this system is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) \right) - V(r) - V(\theta) \quad -(4)$$

where

$$V(r) = -\frac{k}{r} \quad -(5)$$

and where $V(\theta)$ is a theta dependent potential, defining a torque:

$$T_V = -\frac{\partial V(\theta)}{\partial \theta}. \quad -(6)$$

Now consider the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(7)$$

with the Lagrangian (4).

2) It follows that:

$$\frac{\partial \mathcal{L}}{\partial \theta} = Tq = -\frac{\partial V(\theta)}{\partial \theta} \quad -(8)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \frac{d^2\theta}{dt^2} \quad -(9)$$

Therefore:

$$mr^2 \frac{d^2\theta}{dt^2} = Tq = -\frac{\partial V(\theta)}{\partial \theta}$$

$$= \frac{dL_\theta}{dt} \quad -(10)$$

So Eq. (3) follows, Q.E.D.

Here:

$$L_\theta = mr^2 \frac{d\theta}{dt} \quad -(11)$$

and

$$\frac{d\theta}{dt} = \frac{L_\theta(t)}{mr^2} \quad -(12)$$

The total angular momentum is conserved and so L is independent of time and defined by:

$$\frac{d\phi}{dt} = \frac{L}{mr^2} \quad -(13)$$

So

$$\frac{d\theta}{d\phi} = \frac{L_\theta(t)}{L} \quad -(14)$$

3) and:

$$\beta = \frac{L}{L_\theta(t)}\theta \quad -(15)$$

From note 270(3), Eq. (17), the condition for a planar orbit is:

$$\beta = \frac{L}{L_\theta}\theta = \frac{L}{L_\phi}\phi \sin\theta \quad -(16)$$

in which L_θ is a constant of motion:

$$\frac{dL_\theta}{dt} = 0 \quad -(17)$$

It follows from Eq. (16) that:

$$\dot{\beta} = \frac{L}{L_\theta}\dot{\theta} = \frac{L}{L_\phi}\left(\dot{\phi}\sin\theta + \phi\dot{\theta}\cos\theta\right) \quad -(18)$$

whose solution is: $\theta = \frac{\pi}{2}$ $-(19)$

and a planar orbit.

However, when β is defined by Eq. (15):

$$\dot{\beta} = L\frac{d}{dt}\left(\frac{\theta}{L_\theta(t)}\right) \quad -(20)$$

$$= L\left(\frac{L_\theta(t)\dot{\theta} - \theta\dot{L}_\theta(t)}{L_\theta^2(t)}\right)$$

t) So Eq. (18) is no longer true and the orbit is three dimensional:

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(21)$$

where

$$\beta = \left(\frac{L}{L_\phi} \sin \theta \right) \phi \quad -(22)$$

Rotational Langevin Equation

This is:

$$I \frac{d^2\theta}{dt^2} + I \beta \frac{d\theta}{dt} = Tq(\theta) \quad -(23)$$

where

$$Tq(\theta) = - \frac{\partial V(\theta)}{\partial \theta} \quad -(24)$$

Therefore the origin of $V(\theta)$ may be a stochastic, cosmic torque. The solution of eq. (23) is the orientational correlation function:

$$\langle \theta(t) \theta(0) \rangle = \exp(-t/\tau) \quad -(25)$$

where τ is the correlation time. In eq. (23) β is a friction coefficient.

5) So the 3-D orbit (21) reduces to a precessing 2-D orbit if the stochastic background torque becomes smaller and smaller. The relevant quantity that can be measured is the mean square torque $\langle T_q^2(\theta) \rangle$ where $\langle \ \rangle$ is an ensemble average.

<u>Note</u>

It is possible for:

$$L_\theta^2 = \text{constant} \quad -(26)$$

and

$$\frac{d\underline{L}_\theta}{dt} \neq 0 \quad -(27)$$

if

$$\underline{L}_\theta = L_\theta \, \underline{e}_\theta \quad -(28)$$

and

$$\frac{dL_\theta}{dt} = 0, \quad \frac{d\underline{e}_\theta}{dt} \neq 0. \quad -(29)$$

Then

$$L^2 = L_\theta^2 + L_\phi^2 = \text{constant} \quad -(30)$$

<u>QED</u>.