270(4): Three Dimensional Galactic Dynamics:
The Beta Spiral (Hyperbolic)

Consider the three dimensional hyperbolic spiral:

$$\frac{1}{r} = \frac{1}{r_0}\beta \quad -(1)$$

where

$$\beta = \frac{L}{L_\phi}\phi \sin\theta \quad -(2)$$

so

$$\boxed{r = \frac{r_0}{\beta} = \frac{r_0 L_0}{L} \cdot \frac{1}{\phi \sin\theta}} \quad -(3)$$

This function can be plotted in a spherical polar plot.

Its force law from the 3-D Binet equation is:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(4)$$

$$= -\frac{L^2}{mr^3} = -\frac{\partial V(r)}{\partial r}$$

So eq. (3) is produced from an inverse cube law in three dimensions.

The lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) - V(r) \quad -(5)$$

where

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(6)$$

It follows that:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(7)$$

$$\frac{d\theta}{dt} = \frac{L_\theta}{mr^2} \quad -(8)$$

$$\frac{d\phi}{dt} = \frac{L_\phi}{mr^2 \sin\theta} \quad -(9)$$

Therefore:

$$\frac{d\beta}{dt} = \frac{L}{mr_0^2}\beta^2 = \frac{L}{L_\phi}\sin\theta\,\frac{d\phi}{dt} \quad -(10)$$

and

$$\frac{d\phi}{dt} = \frac{L_\phi}{mr_0^2}\sin\theta\,\beta^2, \quad -(11)$$

where

$$\beta = \frac{L}{L_\phi}\phi\sin\theta. \quad -(12)$$

So

$$\frac{d\phi}{dt} = A\phi^2\sin^3\theta \quad -(13)$$

where

$$A = \frac{L^2}{mr_0^2 L_\phi} = \text{constant} \quad -(14)$$

3) It follows that:

$$\frac{1}{\sin^3\theta}\int\frac{d\phi}{\phi^2} = A\int dt \quad -(15)$$

i.e.

$$\boxed{\phi = -\frac{1}{(A\sin^3\theta)\,t}} \quad -(16)$$

The trajectory in Eq. (16) can be animated.

It would model the trajectory in a three dimensional galaxy formed out of a two dimensional whirlpool galaxy.