270(14): Some Basic Concepts in the Plane Polar and Spherical Polar Coordinates

Plane Polar Coordinates

The position vector is:

$$\underset{\sim}{r} = r \underline{e}_r, \quad -(1)$$

The linear velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta, \quad -(2)$$

The acceleration is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(3)$$

The angular momentum is:

$$\underline{L} = m\underset{\sim}{r} \times \underline{v} = mr^2\dot{\theta}\,\underline{e}_r \times \underline{e}_\theta$$

$$= mr^2\dot{\theta}\,\underline{k} \quad -(4)$$

because it has been assumed implicitly by the use of plane polar coordinates that $\underset{\sim}{r}$ and $\underline{v}$ are in the same plane perpendicular to $\underline{L}$.

Using the assumption (4) means that the Coriolis acceleration vanishes for any planar orbit:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(5)$$

so

$$\underset{\sim}{r} \times \underline{a} = \underline{0} \quad -(6)$$

implying

$$\frac{d\underline{L}}{dt} = \underline{Tq} = \underset{\sim}{r} \times \underline{F} = \underline{0}. \quad -(7)$$

Spherical Polar Coordinates

In spherical polars the above restriction are lifted. The position vector is:

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(8)$$

where:

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \quad -(9)$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(10)$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}. \quad -(11)$$

So:

$$\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta \quad -(12)$$

$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r \quad -(13)$$

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi. \quad -(14)$$

The linear velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\sin\theta\,\dot{\phi}\,\underline{e}_\phi \quad -(15)$$

and $\underset{\sim}{r}$ and $\underline{v}$ are not coplanar. The angular momentum is:

$$\underline{L} = m\underset{\sim}{r}\times\underline{v} = mr^2\dot{\theta}\,\underline{e}_r\times\underline{e}_\theta + mr^2\sin\theta\,\dot{\phi}\,\underline{e}_r\times\underline{e}_\phi \quad -(16)$$

$$= mr^2\dot{\theta}\,\underline{e}_\phi - mr^2\sin\theta\,\dot{\phi}\,\underline{e}_\theta$$

$$= mr^2 \left(\dot{\phi} \sin^2\theta \, \underline{k} - (\dot{\theta} \sin\phi + \dot{\phi} \sin\theta \cos\theta \cos\phi) \, \underline{i} + \underline{j} \, (\dot{\theta} \cos\phi - \dot{\phi} \sin\theta \cos\theta \sin\phi) \right)$$

So:

$$L_X = -mr^2 (\dot{\theta} \sin\phi + \dot{\phi} \sin\theta \cos\theta \cos\phi) \quad -(17)$$

$$L_Y = mr^2 (\dot{\theta} \cos\phi - \dot{\phi} \sin\theta \cos\theta \sin\phi) \quad -(18)$$

$$L_Z = mr^2 \dot{\phi} \sin^2\theta \quad -(19)$$

Eqs. (17) - (19) are the same as in UFT 269 and previous notes for UFT 270, QED.

It is clear that there are three Cartesian components of angular momentum in spherical polars, and from eq. (16), two spherical polar components in $\underline{e}_\phi$ and $\underline{e}_\theta$.

The force in spherical polars is:

$$\underline{F} = m \underline{a} = m (a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi) \quad -(20)$$

where:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r \sin^2\theta \, \dot{\phi}^2 \quad -(21)$$

$$a_\theta = 2\dot{r} + r\ddot{\theta} - r \sin\theta \cos\theta \, \dot{\phi}^2 \quad -(22)$$

$$a_\phi = 2\dot{r}\dot{\phi} \sin\theta + 2r\dot{\theta}\dot{\phi} \cos\theta + r \sin\theta \, \ddot{\phi} \quad -(23)$$

4) The force is three dimensional and contains the Coriolis and centripetal components. The usual planar theory of orbits uses the Cartesian:

$$\underline{F} = m \, \underline{\ddot{r}} \quad -(24)$$

and has many terms missing.

From eq. (15):

$$v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \quad -(25)$$

giving the hamiltonian and lagrangian for the inverse square law:

$$H = \frac{1}{2} m v^2 - \frac{k}{r} \quad -(26)$$

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{k}{r} \quad -(27)$$

The velocity can be expressed as:

$$v^2 = \dot{r}^2 + \dot{\beta}^2 r^2 \quad -(28)$$

where

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \quad -(29)$$

and

$$\beta = \tan^{-1}\left(\frac{L}{L_Z} \tan\phi \right) \quad -(30)$$

$$= -\sin^{-1}\left(\frac{L \cos\theta}{\left(L^2 - L_Z^2\right)^{1/2}} \right)$$