269(7): Time Dependence of Angle in a Precessing Elliptical Orbit.

In the first instance consider the precessing planar ellipse:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\phi)} \quad -(1)$$

From the Binet equation the potential for this orbit is:

$$V = -\frac{x^2}{r} + (x^2 - 1)\frac{L^2}{2mr^2} \quad -(2)$$

The Hamiltonian is:

$$H = \frac{p^2}{2m} - \frac{x^2}{r} + (x^2-1)\frac{L^2}{2mr^2} \quad -(3)$$

$$= \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) - \frac{x^2}{r} + \frac{(x^2-1)L^2}{2mr^2}$$

and the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) + \frac{x^2}{r} - \frac{(x^2-1)L^2}{2mr^2} \quad -(4)$$

The Lagrangian variables are r and ϕ, so the Euler Lagrange equations are:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) \quad -(5)$$

and

2)
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right). \quad -(6)$$

Fm Eq. (6):

$$\frac{d}{dt}\left(mr^2 \frac{d\phi}{dt}\right) = 0 \quad -(7)$$

so the following angular momentum is a constant of motion:

$$L = mr^2 \frac{d\phi}{dt}. \quad -(8)$$

It follows that

$$\frac{d\phi}{dt} = \frac{L}{mr^2}\left(1+\epsilon\cos(x\phi)\right)^2 \quad -(9)$$

and

$$\boxed{t = \left(\frac{m\alpha^2}{L}\right)\int \frac{d\phi}{(1+\epsilon\cos(x\phi))^2}} \quad -(10)$$

To evaluate this integral let:

$$\phi_1 = x\phi \quad -(11)$$

so

$$\frac{d\phi_1}{d\phi} = x \quad -(12)$$

and

$$d\phi = \frac{d\phi_1}{x} \quad -(13)$$

3) So:

$$t = \frac{1}{x}\left(\frac{md^2}{L}\right)\int \frac{d\phi_1}{(1+\epsilon\cos\phi_1)^2} \quad -(14)$$

$$= \frac{1}{x}\left(\frac{md^2}{L}\right)\left[\frac{\epsilon\sin\phi_1}{(\epsilon^2-1)(1+\epsilon\cos\phi_1)} - \frac{1}{(\epsilon^2-1)}\int\frac{d\phi_1}{1+\epsilon\cos\phi_1}\right]$$

$$-(15)$$

where:

$$\int\frac{d\phi_1}{1+\epsilon\cos\phi_1} = \frac{2}{(1-\epsilon^2)^{1/2}}\tan^{-1}\left[\frac{(1-\epsilon)\tan(\phi_1/2)}{(1-\epsilon^2)^{1/2}}\right].$$

$$-(16)$$

for the ellipse: $\epsilon^2 < 1. \quad -(17)$

Therefore a plot and animation of t versus ϕ can be made, using:

$$\phi_1 = x\phi \quad -(18)$$

As x gets larger some very intricate trajectories will result. The Eckardt trajectory is

$$x = n = \text{integer} \quad -(19)$$