269(4): The Z Component of the Hamiltonian and its Elliptical Representation

From first principles:

$$L_Z = m r^2 \dot{\phi} \sin^2 \theta \quad -(1)$$

and

$$\dot{\phi} = \frac{L_Z}{m r^2 \sin^2 \theta} \quad -(2)$$

$$:= \frac{L_1}{m r^2}$$

The relevant Hamiltonian is:

$$H_Z = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - \frac{k}{r} \quad -(3)$$

which gives the ellipse:

$$r = \frac{\alpha_1}{1 + \epsilon_1 \cos \phi} \quad -(4)$$

where:

$$\alpha_1 = \frac{L_1^2}{m k} = \frac{L_Z^2}{m k \sin^4 \theta} \quad -(5)$$

and

$$\epsilon_1^2 = 1 + \frac{2 E L_1^2}{m k^2} = 1 + \frac{2 E L_Z^2}{m k^2 \sin^4 \theta} \quad -(6)$$

2) On quantization:

$$L_z \phi = m_\ell \hbar \phi \quad -(7)$$

where
$$m_\ell = \ell, \ell-1, \ldots, -\ell \quad -(8)$$

and
$$\langle L_z^2 \rangle = m_\ell^2 \hbar^2 \quad -(8)$$

In atomic hydrogen:
$$E = -\frac{me^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(9)$$

and
$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(10)$$

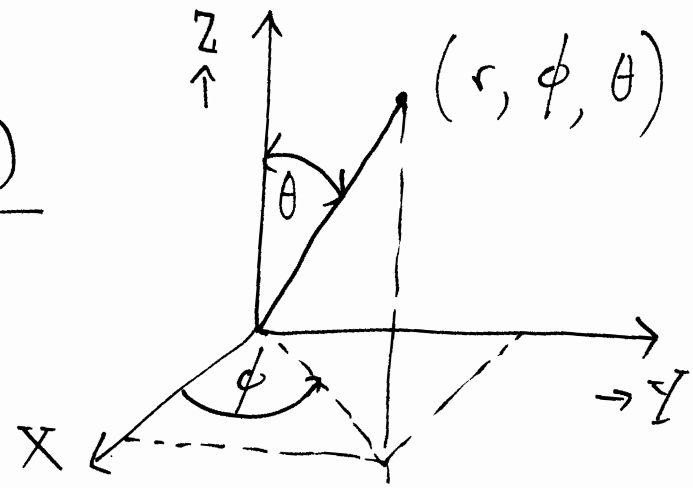
<u>Suggested computer work</u>

The function (4) can be plotted in a three dimensional spherical polar plot in which r is a function of θ and ϕ.

The complete function is:

$$r = \frac{L_z^2}{mk\sin^4\theta}\left[1 + \left(1 + \frac{2EL_z^2}{mk^2\sin^4\theta}\right)^{1/2}\cos\phi\right]^{-1} \quad -(7)$$

Fig. (1)



Here:

$$X = r\sin\theta\cos\phi \quad -(8)$$

$$Y = r\sin\theta\sin\phi \quad -(9)$$

$$Z = r\cos\theta \quad -(10)$$

$$\sin\phi = \frac{Y}{(X^2+Y^2)^{1/2}} \quad -(11)$$

$$\cos\phi = \frac{X}{(X^2+Y^2)^{1/2}} \quad -(12)$$

$$\cos\theta = \frac{Z}{(X^2+Y^2+Z^2)^{1/2}} \quad (13)$$

$$\sin^2\theta = 1 - \cos^2\theta \quad -(14)$$

Therefore r on the left hand side of eq. (7) can be expressed as a function of x, y and z or of θ and ϕ. On the classical level E and L_z can be chosen as parameters. On the quantum level only the eigenvalue of L_z and L^2 operators are observable. It is possible to extend this method to L_x and L_y and this will be done in the next note.
