269(9) : Subsidiary Equation of Motion.

From Eqns (21) and (22) :

$$\beta = \frac{L}{L_\theta} \theta \quad -(1)$$

$$\beta = \frac{L}{L_\phi} \phi \sin\theta \quad -(2)$$

where

$$L_\phi = L_z \quad -(3)$$

$$L_\theta^2 = L^2 - L_z^2 \quad -(4)$$

and

From eq. (1) : $$\dot{\beta} = \frac{L}{L_\theta} \dot{\theta} \quad -(5)$$

and from eq. (2) :

$$\dot{\beta} = \frac{L}{L_\phi} \left(\dot{\phi} \sin\theta + \phi \frac{d}{dt} \sin\theta \right) \quad -(6)$$

where

$$\frac{d}{dt} \sin\theta = \frac{d}{d\theta} \sin\theta \frac{d\theta}{dt} = \dot{\theta} \cos\theta \quad -(7)$$

Adding Eqns (5) and (6) :

$$2\dot{\beta}^2 = \left(\frac{L}{L_\theta}\right)^2 \dot{\theta}^2 + \left(\frac{L}{L_\phi}\right)^2 \left(\dot{\phi} \sin\theta + \phi \dot{\theta} \cos\theta\right)^2 \quad -(8)$$

However,

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \quad - (9)$$

So

$$2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) = \left(\frac{L}{L_\theta}\right)^2 \dot{\theta}^2 + \left(\frac{L}{L_\phi}\right)^2 \left(\dot{\phi}\sin\theta + \phi\dot{\theta}\cos\theta\right)^2 \quad - (10).$$

From eqns. (5) and (6):

$$\frac{L}{L_\theta}\dot{\theta} = \frac{L}{L_\phi}\left(\dot{\phi}\sin\theta + \phi\dot{\theta}\cos\theta\right) \quad - (11)$$

where L/L_θ and L/L_ϕ are constants. From the Lagrangian analysis:

$$\dot{\theta} = \frac{L_\theta}{mr^2} \quad - (12)$$

$$\dot{\phi} = \frac{L_\phi}{mr^2\sin\theta} \quad - (13)$$

$$\dot{\beta} = \frac{L}{mr^2} \quad - (14)$$

so

$$\frac{d\theta}{d\phi} = \frac{L_\theta}{L_\phi}\sin\theta \quad - (15)$$

and

$$\boxed{\theta = \frac{L_\theta}{L_\phi}\,\phi\sin\theta} \quad - (16)$$

3) Eq. (16) is a subsidiary equation of motion.

It implies that:

$$\dot{\theta} = \frac{L_\theta}{L_\phi}\left(\dot{\phi}\sin\theta + \theta\dot{\phi}\cos\theta\right) \quad -(17)$$

and this is eq. (11), Q.E.D.

From Eqs. (9) and (16):

$$\boxed{\dot{\beta}^2 = \dot{\theta}^2\left(1+\left(\frac{L_\theta}{L_\phi}\right)^2\right)} \quad -(18)$$

Results

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(19)$$

where

$$\beta = \frac{L}{L_\theta}\theta = \frac{L}{L_\phi}\sin\theta\,\phi \quad -(20)$$

so:

$$\theta = \frac{L_\theta}{L_\phi}\phi\sin\theta \quad -(21)$$

$$\dot{\beta}^2 = \dot{\theta}^2\left(1+\left(\frac{L_\theta}{L_\phi}\right)^2\right) \quad -(22)$$

$$L^2 = L_\theta^2 + L_\phi^2 = L_\theta^2 + L_Z^2 \quad -(23)$$

$$L_\theta^2 = L^2 - L_Z^2 \quad -(24)$$

4)

These are results of planetary motion in three dimensions, described by spherical polar coordinates.

The orbit is given by eq. (19) and is a precessing ellipse in both θ and ϕ. The precession factor for θ is L / L_θ and the precession factor for ϕ is $(L / L_\phi)\sin\theta\,\phi$!

The static ellipse occurs when either:

$$n = \frac{L}{L_\theta} = 1, 2, 3, \ldots \quad -(25)$$

or

$$n = \frac{L}{L_\phi}\sin\theta = 1, 2, 3, \ldots \quad -(26).$$

The usual theory of orbits is due to Kepler, Hooke and Newton, and is:

$$r = \frac{\alpha}{1 + \epsilon\cos\phi} \quad -(27)$$

in a plane. The 3-D equivalent of this orbit is:

$$r = \frac{\alpha}{1 + \epsilon\cos\left(\frac{L}{L_\phi}\sin\theta\,\phi\right)} \quad -(28)$$

Eq. (28) reduces to Eq. (27) when:

$$\theta = \frac{\pi}{2}, \quad L = L_\phi, \quad L_\theta = 0 \quad -(29)$$

5) Suggested Graphical Work

Graph eq. (28) as a spherical polar plot.

It can also be animated using:

$$x = \frac{L}{L_\phi} \sin\theta = \frac{L}{L_Z} \sin\theta \quad -(30)$$

where $\frac{L}{L_Z}$ = constant of motion

Here:

$$\alpha = \frac{L^2}{mk} \quad -(31)$$

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(32)$$

The total energy E is observable definitely by semi major axis of the ellipse:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2E} \quad -(33)$$

For the inverse square law:

$$k = mMG \quad -(34)$$
