269(5) The $L^2 - L_Z^2$ Hamiltonian and its Elliptical Representation.

Start with the fundamental definitions:

$$L_Z = mr^2 \sin^2\theta \, \dot{\phi} \quad -(1)$$

and:

$$L^2 = m^2 r^4 \left(\dot{\phi}^2 \sin^4\theta + \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta \right)$$

$$= L_Z^2 + L_X^2 + L_Y^2 \quad -(2)$$

So:

$$L_X^2 + L_Y^2 = L^2 - L_Z^2 \quad -(3)$$

$$= m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta \right).$$

From eq. (1):

$$\dot{\phi} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(4)$$

so

$$L_X^2 + L_Y^2 = m^2 r^4 \dot{\theta}^2 + \frac{L_Z^2}{\tan^2\theta} \quad -(5)$$

$$= m^2 r^4 \dot{\theta}^2 + L_Z^2 \cot^2\theta$$

Consider the hamiltonian:

$$H = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{2mr^2} - \frac{k}{r}. \quad -(6)$$

Write eq. (6) as:

2)

$$H = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L_Z^2}{2mr^2} + \frac{L^2 - L_Z^2}{2mr^2} - \frac{k}{r} \quad -(7)$$

so that the complete hamiltonian can be analyzed in terms of L_Z^2 and $L^2 - L_Z^2$. The two types of hamiltonian are:

$$H_1 = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L_Z^2}{2mr^2} - \frac{k}{r} \quad -(8)$$

and the second type is:

$$H_2 = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2 - L_Z^2}{2mr^2} - \frac{k}{r} \quad -(9)$$

The type one hamiltonian H_1 was analyzed in Note 269(4). The type two hamiltonian H_2 is:

$$H_2 = \frac{1}{2} m \left(\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2\right) - \frac{k}{r} \quad -(10)$$

where:

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{L_2^2}{m^2 r^4} \quad -(11)$$

where:

$$\begin{aligned} L_2^2 &= L_X^2 + L_Y^2 - L_Z^2 \cot^2\theta \\ &= L^2 - L_Z^2 (1 + \cot^2\theta) \end{aligned} \quad -(12)$$

3) The types are and two ellipses are:

$$r = \frac{\alpha_1}{1+\epsilon_1 \cos\phi} \quad -(13)$$

and

$$r = \frac{\alpha_2}{1+\epsilon_2 \cos\theta} \quad -(14)$$

where:

$$\alpha_1 = \frac{L_1^2}{nk} = \frac{L_z^2}{nk \sin^4\theta}, \quad -(15)$$

$$\epsilon_1^2 = 1 + \frac{2EL_z^2}{nk^2 \sin^4\theta}, \quad -(16)$$

and

$$\alpha_2 = \frac{L_2^2}{nk} = \frac{1}{nk}\left(L^2 - L_z^2\left(1+\cot^2\theta\right)\right) \quad -(17)$$

with:

$$\epsilon_2^2 = 1 + \frac{2E}{nk^2}\left(L^2 - L_z^2\left(1+\cot^2\theta\right)\right) \quad -(18)$$

The functions (13) and (14) can be graphed and analyze to consider overall trajectory:

4) $$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(19)$$

where $$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(20)$$

1) In general the inverse square law of attraction generates these classical three dimensional orbits in cosmology or in atoms and molecules.

2) There is no apparent way of graphing Eq. (19), but Eqs. (13) and (14) can be graphed.

3) The precessing ellipses are:

$$r = \frac{\alpha_1}{1+\epsilon_1\cos(x\phi)} \quad -(21)$$

and $$r = \frac{\alpha_2}{1+\epsilon_2\cos(x\theta)} \quad -(22)$$