269(8): Complete Solution of the β Ellipse

The β ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(1)$$

and corresponds to the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) + \frac{k}{r} \quad -(2)$$

where

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(3)$$

The Euler Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\beta}}\right) \quad -(4)$$

gives the conserved angular momentum:

$$L = mr^2\dot{\beta} \quad -(5)$$

so

$$\frac{d\beta}{dt} = \frac{L}{mr^2} = \frac{L}{m\alpha^2}\left(1+\epsilon\cos\beta\right)^2 \quad -(6)$$

and

$$t = \frac{m\alpha^2}{L}\int\frac{d\beta}{(1+\epsilon\cos\beta)^2} \quad -(7)$$

which has the same analytical solution as in the previous notes.

d) From first principles:

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$= m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta + \dot{\phi}^2 \sin^4\theta\right)$$

$$= m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

$$= m^2 r^4 \dot{\beta}^2 \qquad -(12)$$

and this checks that the definition of β is correct.

From the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \qquad -(13)$$

it follows that:

$$L_\theta = m r^2 \dot{\theta} = \text{constant} \qquad -(14)$$

So

$$L^2 = L_\theta^2 + m^2 r^4 \dot{\phi}^2 \sin^2\theta \qquad -(15)$$

It follows that

$$L_\phi = m r^2 \dot{\phi} \sin\theta \qquad -(16)$$

is a constant of motion, and:

$$L^2 = L_\theta^2 + L_\phi^2 \qquad -(17)$$

Therefore it follows that:

$$\frac{d\beta}{dt} = \frac{L}{mr^2}, \quad \frac{d\theta}{dt} = \frac{L_\theta}{mr^2}, \quad \frac{d\phi}{dt} = \frac{L_\phi}{mr^2 \sin\theta} \quad -(18)$$

and

$$\frac{d\beta}{d\theta} = \frac{L}{L_\theta} \quad -(19)$$

$$\frac{d\beta}{d\phi} = \frac{L}{L_\phi} \sin\theta \quad -(20)$$

So

$$\beta = \frac{L}{L_\theta} \theta \quad -(21)$$

and

$$\beta = \frac{L}{L_\phi} \phi \sin\theta \quad -(22)$$

and the integration of Eq. (3) has been achieved.

Therefore the β ellipse (1) can be expressed as:

$$\boxed{r = \frac{\alpha}{1+\epsilon\cos\left(\frac{L}{L_\theta}\theta\right)} = \frac{\alpha}{1+\epsilon\cos\left(\frac{L}{L_\phi}\sin\theta\,\phi\right)}} \quad -(23)$$

and these are precessing ellipses.

In Eq. (23):

$$\alpha = \frac{L^2}{mk} \quad -(24)$$

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(25)$$

where $$L^2 = L_\theta^2 + L_\phi^2 \quad -(26)$$

From eqs. (18):

$$t = \frac{m\alpha^2}{L}\int \frac{d\beta}{(1+\epsilon\cos\beta)^2} \quad -(27)$$

where $$\beta = \frac{L}{L_\theta}\theta = \frac{L}{L_\phi}\phi\sin\theta. \quad -(28)$$

so: $$t = \frac{m\alpha^2}{L_\theta}\int \frac{d\theta}{\left(1+\epsilon\cos\left(\frac{L}{L_\theta}\theta\right)\right)}$$

$$= \frac{m\alpha^2\sin\theta}{L_\phi}\int \frac{d\phi}{\left(1+\epsilon\cos\left(\frac{L}{L_\phi}\sin\theta\,\phi\right)\right)}$$

$$-(29)$$