269(2): Constants of Motion of Angular Momentum in Spherical Polar Coordinates

The angular momentum in three dimensions is:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(1)$$

and is conserved, i.e.

$$\frac{d\underline{L}}{dt} = 0. \quad -(2)$$

In spherical polar coordinates:

$$\underline{r} = r\sin\theta\cos\phi\,\underline{i} + r\sin\theta\sin\phi\,\underline{j} + r\cos\theta\,\underline{k} \quad -(3)$$

and

$$\underline{p} = m\Big((\dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})\underline{i}$$
$$+ (\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\underline{j}$$
$$+ (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\underline{k}\Big) \quad -(4)$$

So:

$$\underline{r} \times \underline{p} = m\Big[\underline{i}\,\Big(r\sin\theta\cos\phi(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})$$
$$- r\cos\theta(\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\Big)$$
$$+ \underline{j}\,\Big(r\cos\theta(\dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})$$
$$- (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\,r\sin\theta\cos\phi\Big)$$
$$+ \underline{k}\,\Big(r\sin\theta\cos\phi(\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta}$$
$$+ r\sin\theta\cos\phi\,\dot{\phi}) - r\sin\theta\cos\phi(\dot{r}\sin\theta\cos\phi$$
$$+ r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})\Big) \quad -(5)$$

2) and $L = L_x \underline{i} + L_y \underline{j} + L_z \underline{k}$ — (6)

so $$\frac{dL_x}{dt} = \frac{dL_y}{dt} = \frac{dL_z}{dt} = 0 \quad \text{— (7)}$$

So L_x, L_y and L_z are constants of motion.

For example:

$$\underline{L}_Z = L_z \underline{k} = m\underline{k}\big(r\dot{r}\sin^2\theta\sin\phi\cos\phi + r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2(\sin\theta\cos\theta\sin\phi\cos\phi)\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\cos\phi\,\dot{\theta} - r\dot{r}\sin^2\theta\sin\phi\cos\phi + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}\big)$$

$$= m r^2 \sin^2\theta\,\dot{\phi}\,\underline{k} \quad \text{— (8)}$$

This is the same result as in note 269(1), using a Lagrangian method.

Similarly:

$$\cancel{L_x = m\big(r\dot{r}\sin\theta\cos\theta(\cos\phi - \sin\phi) - r^2\dot{\theta}(\sin^2\theta\cos\phi + \cos^2\theta\sin\phi) - r^2\dot{\phi}\sin\theta\cos\theta\big) \quad \text{— (9)}}$$

$$\cancel{L_y = m(}$$

4) $L_x = -mr^2\left(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi\right)$ — (9)

$L_y = mr^2\left(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi\right)$ — (10)

If $\theta = \pi/2$, $\sin\theta = 1$, $\cos\theta = 0$, &c:

$L_x = -mr^2\dot{\theta}\sin\phi$ — (11)

$L_y = mr^2\dot{\theta}\cos\phi$ — (12)

$L_z = mr^2\dot{\phi}$ — (13)

the square of the angular momentum is: — (14).

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$= m^2r^4\left(\sin^4\theta\,\dot{\phi}^2 + \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\cos^2\theta\right)$$

$$= m^2r^4\left(\sin^2\theta\,\dot{\phi}^2 + \dot{\theta}^2\right)$$

The Hamiltonian is:

$$H = \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right) - \frac{k}{r}$$

$$= \frac{1}{2}m\left(\dot{r}^2 + \frac{L^2r^2}{m^2r^4}\right) - \frac{k}{r}$$

$$= \frac{1}{2}m\dot{r} + \frac{L^2}{2mr^2} - \frac{k}{r} \quad -(15)$$

4) Conclusions

The Lagrangian in three dimensions is the same as that in two dimensions with:

$$\dot{\phi}^2 \rightarrow \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(16)$$

There are two terms in this analysis that are missing from an analysis with only r and ϕ as Lagrangian variables. The three constants of motion are given by eqs. (8), (9) and (10).

If one defines:

$$\dot{\chi}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(17)$$

the relevant ellipse is:

$$r = \frac{\alpha}{1 + \epsilon \cos\chi} \quad -(18)$$