267(1): Sommerfeld Type Theory of Orbital Precession

The hamiltonian for this theory is:

$$H = E = (\gamma - 1) mc^2 - \frac{mMG}{r} \quad -(1)$$

where the relativistic kinetic energy is:

$$T = (\gamma - 1) mc^2 \quad -(2)$$

where

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

is the Lorentz factor. The orbital velocity v is defined by:

$$ds^2 = c^2 d\tau^2 = (c^2 - v^2) dt^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(4)$$

This is a Minkowski metric in plane polar coordinates. This is in which

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \quad -(5)$$

is the non-relativistic orbital velocity. The latter is defined by:

$$H = E = \frac{1}{2} mv^2 - \frac{mMG}{r} \quad -(6)$$

From eq. (6) it is well known that:

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(7)$$

where a is the semi major axis of the ellipse:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(8)$$

Fm eqs. (1) and (7):

$$E = \left[\left(1 - \frac{MG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)\right)^{-1/2} - 1\right]mc^2 - \frac{mMG}{r} \quad -(9)$$

$$= \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 - \frac{mMG}{r}.$$

Now expand the Lorentz factor as follows:

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 - \frac{1}{2}\left(\frac{v^2}{c^2}\right) + \frac{3}{8}\left(\frac{v^4}{c^4}\right) + \frac{5}{16}\left(\frac{v^6}{c^6}\right) + \cdots \quad -(10)$$

It follows that:

$$E = \frac{1}{2}mv^2 - \frac{mMG}{r} + \frac{3}{8}\frac{mv^4}{c^2} + \cdots \quad -(11)$$

where

$$v^4 = M^2G^2\left(\frac{2}{r} - \frac{1}{a}\right)^2. \quad -(12)$$

3) So:

$$E = \frac{1}{2}mv^2 - \frac{mMG}{r} + \frac{3}{8}\frac{mM^2G^2}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)^2$$

$$= T + V \quad -(13)$$

where

$$V = -\frac{mMG}{r} + \frac{3}{8}\frac{mM^2G^2}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)^2 \quad -(14)$$

is the potential energy. The latter can be expanded as:

$$V = -\frac{mMG}{r} + \frac{3}{8}\frac{mM^2G^2}{c^2}\left(\frac{4}{r^2} - \frac{4}{ar} + \frac{1}{a^2}\right)$$

$$= -\frac{1}{r}\left(mMG + \frac{3}{2}\frac{mM^2G^2}{ac^2}\right) + \frac{3}{2}\frac{mM^2G^2}{c^2r^2}$$

$$+ \frac{3}{8}\frac{mM^2G^2}{a^2c^2} \quad -(15)$$

The force is:

$$F = -\frac{\partial U}{\partial r} = -\frac{1}{r^2}\left(mMG\left(1 + \frac{3MG}{2ac^2}\right)\right)$$

$$+ \frac{mMG}{r^3}\left(\frac{3MG}{c^2}\right) \quad -(16)$$

This is the force law of a precessing

4) If this model is applied to the Sommerfeld atom then it is known that the orbitals are nearly circular, so

$$v^2 = \frac{2MG}{r} \quad - (17)$$

which implies

$$\frac{1}{a} \to 0 \ . \quad - (18)$$

Therefore eq. (16) becomes:

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} + \left(\frac{3MG}{c^2}\right)\frac{mMG}{r^3} \quad - (19)$$

The force law of a precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad - (20)$$

is

$$F = -x^2\frac{mMG}{r^2} + \frac{(x^2-1)L^2}{mr^3} \quad - (21)$$

If it is assumed that:

$$x \sim 1 \quad - (22)$$

Eq. (21) can be approximated with:

$$F \sim \frac{mMG}{r^2} + \frac{(x^2-1)L^2}{mr^3} \quad - (23)$$

Comparing eqs. (19) and (23):

$$\frac{(x^2-1)L^2}{mr^3} = \frac{3mG^2m^2}{c^2r^3} \quad -(24)$$

and

$$x^2 = 1 + \left(\frac{3mG}{c^2}\right)\frac{m^2mG}{L^2} \quad -(25)$$

$$= 1 + \frac{3mG}{c^2\alpha}$$

So

$$x = \left(1 + \frac{3mG}{c^2\alpha}\right)^{1/2} = \left(1 + \frac{3mG}{2c^2\alpha}\right). \quad -(26)$$

This theory describes the gravitational interaction between the electron and proton in the near circular approximation. The approximate result (26) is similar to that of planetary precession:

$$x_p = 1 + \frac{3mG}{c^2\alpha} \quad -(27)$$

but there is a factor of a half in eq.(26). This method proves that the Sommerfeld

6) Lagrangian (1) produces a precessing ellipse. It is not suitable for planetary precession because the velocity of planets in orbit is not relativistic. Also, it is known that the orbital precession of planets is the Thomas precession, and to incorporate the latter correctly in spin orbit coupling requires the fermion equation, the Sommerfeld equation cannot produce the Thomas factor.

In the near circular approximation it is possible to add more terms to Eq. (11):

$$E = \frac{1}{2} mv^2 - \frac{mMG}{r} + \frac{3}{8}\frac{mv^4}{c^2} + \frac{5}{16}\frac{mv^6}{c^4} + \dots \quad -(28)$$

giving the potential:

$$V = -\frac{mMG}{r} + \frac{3}{2}\, m \frac{M^2 G^2}{c^2 r^2} + \frac{5}{16}\frac{m \cdot 8}{c^4}\frac{M^3 G^3}{r^3} + \dots \quad -(29)$$

but this leads to a force law that is not that of the precessing ellipse (20).