267(2): Relation between Bohr and Schroedinger Quantization in ECE Theory.

On the classical level both theories are described by

$$\frac{p^2}{2m} = \frac{k}{r} + E \quad -(1)$$

where p is the momentum of the electron, m its mass, r is the distance between the electron and proton, and E is the total energy. Here

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(2)$$

where $-e$ is the charge on the electron and ϵ_0 the vacuum permittivity. Both theories are described by the same force equation:

$$F = m(\ddot{r} - r\dot{\theta}^2) = -\frac{L}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)$$

$$= -\frac{\partial U}{\partial r} = -\frac{e^2}{4\pi\epsilon_0 r^2} \quad -(3)$$

where

$$\omega = \dot{\theta} = \frac{L}{mr^2} \quad -(4)$$

Here L is the conserved total angular momentum, and where ω is the angular velocity, the spin connection of ECE theory. Therefore:

$$m\frac{d^2r}{dt^2} = \frac{L^2}{mr^3} - \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(5)$$

The solution of eq. (3) is:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad - (6)$$

for both Bohr and Schroedinger quantization, because they originate in the same classical theory. In Eq. (6):

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2E\alpha}{k} \quad - (7)$$

Bohr quantization is defined by:

$$L = \hbar n \quad - (8)$$

where

$$n = 0, 1, 2, \ldots \quad - (9)$$

The Bohr radius is:

$$r = \alpha = \frac{n^2 \hbar^2}{mk} \quad - (10)$$

and the Bohr energy levels are defined by:

$$\epsilon = 0 \quad - (11)$$

i.e

$$E = -\frac{k}{2\alpha} = -\frac{mk^2}{2n^2\hbar^2} \quad - (12)$$

Schroedinger quantization is defined by:

$$\underline{p} = -i\hbar \underline{\nabla} \quad - (13)$$

So:

$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} \quad -(14)$$

Therefore eq. (1) becomes:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial r^2} = \left(\frac{k}{r}+E\right)\phi \quad -(15)$$

For motion in a plane:

$$X = r\cos\theta,\quad Y = r\sin\theta \quad -(16)$$

and

$$\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \quad -(17).$$

By the chain rule:

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial X} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial X} \quad -(18)$$

$$\frac{\partial f}{\partial Y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial Y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial Y} \quad -(19)$$

in which:

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial \theta}d\theta \quad -(20)$$

$$dY = \frac{\partial Y}{\partial r}dr + \frac{\partial Y}{\partial \theta}d\theta \quad -(21)$$

so

$$dX = \cos\theta\, dr - r\sin\theta\, d\theta \quad -(22)$$

$$dY = \sin\theta\, dr + r\cos\theta\, d\theta \quad -(23)$$

From eqns. (22) and (23):

4)

$$dr = \cos\theta \, dX + \sin\theta \, dY \quad -(24)$$

$$d\theta = -\frac{\sin\theta}{r} dX + \frac{\cos\theta}{r} dY \quad -(25)$$

Fm eqs. (16):

$$\frac{\partial f}{\partial X} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial \theta} \quad -(26)$$

$$\frac{\partial f}{\partial Y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r} \frac{\partial f}{\partial \theta} \quad -(27)$$

Therefore: $$\frac{\partial^2 f}{\partial X^2} + \frac{\partial^2 f}{\partial Y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial r} \quad -(28)$$

For motion on a ring of fixed radius r_B:

$$\frac{\partial^2 f}{\partial X^2} + \frac{\partial^2 f}{\partial Y^2} = \frac{1}{r_B^2} \frac{\partial^2 f}{\partial \theta^2} \quad -(29)$$

This motion corresponds to a constant total energy:

$$E = T = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m r^2 \omega^2 = \frac{1}{2} m v_\theta^2 \quad -(30)$$

Therefore it quantizes to:

3)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial r^2} = E\phi \quad -(31)$$

i.e

$$-\frac{\hbar^2}{2m r_B^2}\frac{\partial^2\phi}{\partial\theta^2} = E\phi \quad -(32)$$

or

$$\frac{\partial^2\phi}{\partial\theta^2} = -\left(\frac{2m r_B^2 E}{\hbar^2}\right)\phi \quad -(33)$$

The quantum number is m_l, type of quantization is denoted

$$\boxed{m_l^2 = \frac{2m r_B^2 E}{\hbar^2}} \quad -(34)$$

In the Bohr theory, eq. (15) becomes:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\theta^2} = k r_B - \frac{k r_B}{2} \quad -(35)$$

where r_B is the Bohr radius:

$$r_B = \frac{\hbar}{mc}\frac{n^2}{\alpha_f} \quad -(36)$$

where the fine structure constant is:

$$\alpha_f = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(37)$$

6) So $$\boxed{\frac{\partial^2 \phi}{\partial \theta^2} = -\frac{k n^2}{I c \omega_f}\phi = -n^2 \phi} \quad -(38)$$

Comparing eqs. (33) and (38):

$$\boxed{n = m_\ell} \quad -(39)$$

The two quantization schemes are the same provided n of the Bohr theory is m_ℓ of the quantized particle on a ring.

Both theories are special cases of the elliptical orbit (6) when the ellipticity ϵ vanishes.

The wavefunction corresponding to this type of quantization is well known to be:

$$\phi(\theta) = \left(\frac{1}{2\pi}\right)^{1/2} e^{i m_\ell \theta}, \quad -(40)$$

so is the Bohr atom:

$$\phi(\theta) = \left(\frac{1}{2\pi}\right)^{1/2} e^{i n \theta} \quad -(41)$$

In this type of quantization an integral number of waves appears on a circle.

1) This result is similar to Eckardt quantization in UFT 266, in which

$$r = \frac{\alpha}{1+\epsilon\cos(m\theta)} \quad -(42)$$

which means that:

$$\alpha = \frac{r_0}{m-1} \quad -(43)$$

where r_0 is a constant, and where m is an integer. If Eq. (43) is to be identified with the Bohr radius then

$$r_0 = r_B = (m-1)\alpha = n\alpha \quad -(44)$$

so

$$n = m_\ell = m-1 \quad -(45)$$

As shown in UFT 266 eq. (42) produces waves superimposed on an ellipse. In Bohr and particle on a ring quantization the ellipse approaches a circle. The conclusion is that Eckardt theory and quantization leads to Bohr / Schroedinger quantization, from x theory with $x = m$.