267(6): Three Dimensional Orbits

The lagrangian is:

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{k}{r} = T - V \quad -(1)$$

where

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) \quad -(2)$$

in the notation of previous UFT papers. The three Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(4)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(5)$$

From eq. (3):

$$m \left(\ddot{r} - r \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right) \right) = -\frac{k}{r^2} \quad -(6)$$

From eq. (4):

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt} m r^2 \dot{\theta} = 0. \quad -(7)$$

From eq. (5):

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt} m r^2 \sin^2\theta \, \dot{\phi} = 0 \quad -(8)$$

2)

So the conserved angular momenta are:

$$L_1 = mr^2\dot{\theta} \quad -(9)$$

and

$$L_2 = mr^2\sin^2\theta\,\dot{\phi} \quad -(10)$$

It follows that:

$$m\frac{d^2r}{dt^2} = -\frac{k}{r^2} + \frac{L^2}{mr^3} \quad -(11)$$

where

$$L^2 = L_1^2 + L_2^2 \quad -(12)$$

The format of the Leibniz equation (11) is not changed by going from two to three dimensions, but the total angular momentum is defined by eq. (12).

It also follows that:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{m}{L_1\dot{\theta}}\,\ddot{r} \quad -(13)$$

and

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = -\frac{m}{L_2\dot{\phi}}\,\ddot{r} \quad -(14)$$

so there are two Binet equations:

$$-\frac{L_1^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{L^2}{m^2r^3} = -\frac{k}{r}$$

$$-(15)$$

and:

$$-\frac{L_2^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)-\frac{L^2}{m^2r^3}=-\frac{k}{r} \quad -(16)$$

in which:

$$m\frac{d^2r}{dt^2}=-\frac{L_1^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(17)$$

$$=-\frac{L_2^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)$$

There are two ellipses:

$$r=\frac{\alpha_1}{1+\epsilon_1\cos\theta} \quad -(18)$$

and

$$r=\frac{\alpha_2}{1+\epsilon_2\cos\phi} \quad -(19)$$

giving:

$$m\frac{d^2r}{dt^2}=\frac{L_1^2}{mr^2}\left(\frac{1}{r}-\frac{1}{\alpha_1}\right) \quad -(20)$$

$$=\frac{L_2^2}{mr^2}\left(\frac{1}{r}-\frac{1}{\alpha_2}\right)$$

i.e two Leibniz equations