265(5): Further Details of the Derivation of the Planetary Precession from the Thomas Precession.

Consider the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

in plane polar coordinates, where τ is the proper time. The Thomas precession is defined as:

$$\theta' = \theta + \omega t, \quad -(2)$$

$$d\theta' = d\theta + \omega dt. \quad -(3)$$

where ω is an angular velocity. Therefore:

$$(d\theta')^2 = (d\theta + \omega dt)^2 \quad -(4)$$

$$= d\theta^2 + 2\omega d\theta dt + \omega^2 dt^2.$$

It follows that under the Thomas precession:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2(d\theta^2 + 2\omega d\theta dt + \omega^2 dt^2)$$

$$= (c^2 - r^2\omega^2) dt^2 - dr^2 - r^2 d\theta^2 - 2\omega r^2 d\theta dt \quad -(5)$$

The Thomas precession is defined by the velocity:

$$v_\theta = r\omega \quad -(6)$$

so

$$ds^2 = \left(1 - \frac{v_\theta^2}{c^2}\right) c^2 dt^2 - dr^2 - r^2 d\theta^2 - 2\omega r^2 d\theta dt \quad -(7)$$

Now use:

$$\omega = \frac{d\theta}{dt}, \quad v_\theta = \omega r \quad -(8)$$

2) So

$$d\theta = \omega\, dt = \frac{v_\theta}{r}\, dt. \quad -(9)$$

It follows that:

$$ds^2 = \left(1 - \frac{3v_\theta^2}{c^2}\right) c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(10)$$

The orbital velocity is defined by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2, \quad -(11)$$

So

$$c^2 d\tau^2 = \left(1 - \frac{3v_\theta^2}{c^2}\right) c^2 dt^2 - v^2 dt^2 \quad -(12)$$

i.e.

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{3v_\theta^2}{c^2} - \frac{v^2}{c^2} \quad -(13)$$

The Lorentz factor is modified to:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{3v_\theta^2}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \quad -(14)$$

$$\sim \frac{v^2}{c^2} + \frac{3v_\theta^2}{c^2}$$

if $v \ll c$ and $v_\theta \ll c$ $\quad -(15)$

3) The total linear velocity is:

$$\underline{v} = \underset{\sim}{v_r} + \underset{\sim}{\omega} \times \underset{\sim}{r} \quad -(16)$$

where

$$\underline{v}_r = \frac{dr}{dt} \underline{e}_r \quad -(17)$$

so the total kinetic energy is:

$$T = \frac{1}{2} m \left(v_r^2 + \underset{\sim}{\omega} \times \underset{\sim}{r} \cdot \underset{\sim}{\omega} \times \underset{\sim}{r} \right)$$

$$= \frac{1}{2} m \left(v_r^2 + \omega^2 r^2 \right) \quad -(18)$$

if

$$\underset{\sim}{\omega} \perp \underset{\sim}{r} \quad -(19)$$

The radial kinetic energy is:

$$T_r = \frac{1}{2} m v^2 \quad -(20)$$

and the angular kinetic energy is:

$$T_\theta = \frac{1}{2} m \omega^2 r^2 = \frac{1}{2} I \omega^2 \quad -(21)$$

where the moment of inertia is:

$$I = m r^2. \quad -(22)$$

For a constant ω:

$$T = \frac{1}{2} m \omega^2 r^2 = m \int \omega^2 r \, dr \quad -(23)$$

which is the integral of the centrifugal force.

In generalizing the Thomas precession, the only

4) velocity considered is:

$$\underline{V_\theta} = \underset{\sim}{\omega} \times \underset{\sim}{r} \quad -(24)$$

i.e

$$= \omega r \, \underline{e_\theta}$$

$$V_\theta = \omega r \quad -(25)$$

So

$$\underset{\sim}{F} = m\omega^2 r \, \underline{e}_r = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(26)$$

and

$$\int F dr = \frac{1}{2} m V_\theta^2 = \frac{mMG}{r} \quad -(27)$$

So

$$\boxed{V_\theta^2 = \frac{2MG}{r}} \quad -(28)$$

The infinitesimal line element (12) is therefore:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{6MG}{c^2 r}\right) c^2 dt^2 - v^2 dt^2 \quad -(29)$$

The additional time dilation due to the Thomas precession is:

$$\frac{dt}{d\tau} = \left(1 - \frac{6MG}{c^2 r}\right)^{-1/2}$$

$$\sim \frac{3MG}{c^2 r} \quad -(30)$$

) if $\frac{MG}{c^2} << r$ — (31)

The definition (24) of $\underline{V_\theta}$ used in the Thomas precession corresponds to a turning point of any orbit:

$$\underline{V_r} = \underline{0} \quad -(32)$$

Consider now the effect of the Thomas precession on an elliptical orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(33)$$

Its turning point occurs at:

$$r = \alpha = a(1-\epsilon^2) \quad -(34)$$

where α is the half right latitude, ϵ is the eccentricity, and a the semi major axis. The minimum value of r in eq. (33) is the perihelion, when m is closest to M. The perihelion is a turning point. So at the perihelion: — (35)

$$\boxed{\frac{dt}{d\tau} = \frac{3MG}{c^2\alpha} = \frac{3MG}{c^2 a(1-\epsilon^2)}}$$

which is the planetary precession of the perihelion,

Q.E.D.

The planetary precession is therefore the Thomas precession. Its effect is:

$$\theta \rightarrow \theta(1+x) \quad -(36)$$

where

$$x = \gamma_T = \frac{dt}{d\tau} = \frac{3MG}{c^2\alpha} \quad -(37)$$

Therefore:

$$\boxed{dt \rightarrow \gamma_T \, d\tau} \quad -(38)$$

The precession changes the spin connection or angular velocity to:

$$\Omega := \frac{d\theta}{d\tau} = \gamma_T \frac{d\theta}{dt} \quad -(39)$$

This can be used as the definition of the relativistic angular velocity:

$$\Omega = \gamma_T \, \omega. \quad -(40)$$

Conclusion

The rotation of the Minkowski metric produces the universal planetary precession. It is not due to the incorrect Einstein equation.
