65(3): Fundamental Origin of Orbital Precession in the Thomas Precession.

Consider the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

The Thomas precession is defined by:

$$\theta' = \theta + \omega t \quad -(2)$$

where

$$v_\theta = \omega r \quad -(3)$$

where ω is the angular velocity and v_θ the angular part of:

$$v^2 = v_r^2 + v_\theta^2 \quad -(4)$$

Therefore:

$$d\theta'^2 = (d\theta + \omega dt)^2 \quad -(5)$$

$$= d\theta^2 + 2\omega\, d\theta\, dt + \omega^2 dt^2$$

It follows that:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2(2\omega\, d\theta\, dt + \omega^2 dt^2) \quad -(6)$$

$$= (c^2 - r^2\omega^2)\, dt^2 - 2r^2\omega\, d\theta\, dt - dr^2 - r^2 d\theta^2$$

Now use:

2)

$$\omega = \frac{d\theta}{dt}, \quad d\theta = \omega\, dt \quad -(7)$$

Eq. (6) becomes:

$$ds^2 = \left(1 - \frac{v_\theta^2}{c^2}\right) c^2 dt^2 - 2 v_\theta^2 dt^2 - dr^2 - r^2 d\theta^2$$

$$= \left(1 - 3\frac{v_\theta^2}{c^2}\right) c^2 dt^2 - dr^2 - r^2 d\theta^2$$

$$= c^2 d\tau^2 \quad -(8)$$

Now use:

$$v^2 dt^2 = dr^2 + r^2 d\theta^2 \quad -(9)$$

so

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - 3\frac{v_\theta^2}{c^2} - \frac{v^2}{c^2}\right) \quad -(10)$$

The Thomas precession changes the Lorentz factor to:

$$\boxed{\gamma = \left(1 - \frac{v^2}{c^2} - 3\frac{v_\theta^2}{c^2}\right)^{-1/2}} \quad -(11)$$

where

$$v^2 = v_r^2 + v_\theta^2. \quad -(12)$$

For a conical section orbit:

3)

$$v^2 = mG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(13)$$

where for an ellipse:

$$\alpha = a(1-\epsilon^2) \quad -(14)$$

The Thomas precession uses eq. (3), which is the definition of v_θ for a circle in which:

$$\epsilon = 0, \quad r = \alpha \quad -(15)$$

so for a circle:

$$v_\theta^2 = \frac{mG}{r} = \frac{mG}{\alpha} \quad -(16)$$

Therefore the Thomas precession introduces the time dilatation:

$$\frac{d\tau}{dt} = \left(1 - \frac{3mG}{c^2\alpha}\right)^{1/2} \quad -(17)$$

Therefore this is the fundamental origin of the factor

$$r_0 = \frac{3mG}{c^2} \quad -(18)$$

We have

$$v_\theta^2 = \omega^2 r^2 = \frac{3mG}{\alpha} \quad -(19)$$

and

$$\omega^2 = \frac{3mG}{\alpha r^2} = \frac{3mG}{\alpha^2} \quad -(20)$$

In the Thomas precession the rotation of

t) the frame at an angular velocity ω produce the precession $\theta \rightarrow x\theta = \left(1 + \frac{\omega t}{\theta}\right)\theta$ - (21)

and from eq. (17):

$$\frac{3mG}{c^2 \alpha} = 1 - \left(\frac{d\tau}{dt}\right)^2 \quad -(22)$$

in the limit of a circular orbit, because the Thomas precession assumes eq. (3).

So if we define the orbital precession for convenience as:

$$x = 1 - \frac{3mG}{c^2 \alpha} \quad -(23)$$

it is found that:

$$\boxed{\left(\frac{d\tau}{dt}\right)^2 = \frac{3mG}{c^2 \alpha}} \quad -(24)$$

The Thomas precession rotates the frame of an ellipse with an angular velocity:

$$\omega = \frac{v_\theta}{r} \quad -(25)$$

and results in the precession (23) in the limit of a circle ($\epsilon \rightarrow 0$). A circular orbit precesses by (24), with $\alpha = r$.

5) For the general orbit:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(26)$$

$$= V_r^2 + V_\theta^2$$

So for the general orbit:

$$V_\theta = \omega r \quad -(27)$$

is also true, with:

$$V_r = \omega \frac{dr}{d\theta} \quad -(28)$$

At the turning point of the orbit the V_r component vanishes and

$$v^2 = V_\theta^2 \quad -(29)$$

and at closest approach:

$$r = \frac{\alpha}{1+\epsilon} = a \quad -(30)$$

At the turning point of the precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(31)$$

then

$$r = \alpha \quad -(32)$$

From eqns. (29) and (32), for the general orbit eq. (24) is also true with:

6)

$$\alpha = a(1-\epsilon^2) \quad -(33)$$

In general the planetary precession is given by the Thomas precession:

$$\boxed{\left(\frac{d\tau}{dt}\right)^2 = \frac{3MG}{c^2 a(1-\epsilon^2)}} \quad -(34)$$

The perihelion is a turning point, so the precession of the perihelion is given by eqn. (34).