263(8): Calculation of Light Deflection due to Gravitation

As shown in previous notes the old Schwarzschild metric is equivalent to:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)}, \quad -(1)$$

where

$$x = 1 + \frac{r_0}{\alpha}, \quad -(2)$$

$$r_0 = 3mG/c^2. \quad -(3)$$

Therefore the light deflection due to gravitation can be calculated using:

$$\Delta\theta = 2\int_{R_0}^{\infty} \frac{d\theta}{dr}\,dr - \pi \quad -(4)$$

From eq. (1):

$$\frac{dr}{d\theta} = \frac{x\epsilon r^2 \sin(x\theta)}{\alpha} \quad -(5)$$

so

$$\Delta\theta = 2\int_{R_0}^{\infty} \frac{\alpha\,dr}{x\epsilon r^2\sin(x\theta)} - \pi \quad -(6)$$

From eq. (1):

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(7)$$

and

$$\cos^2(x\theta) + \sin^2(x\theta) = 1 \quad -(8)$$

so:

2)

$$\sin(x\theta) = \left(\left(1 - \frac{1}{\epsilon^2}\right)\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(9)$$

So

$$\Delta\theta = \frac{2\alpha}{x\epsilon}\int_{R_0}^{\infty} \frac{1}{r^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2} dr - \pi \quad -(10)$$

where

$$\frac{1}{x} = \left(1 + \frac{r_0}{\alpha}\right)^{-1} \quad -(11)$$

$$\sim \left(1 - \frac{r_0}{\alpha}\right)$$

for

$$r_0 << \alpha \quad -(12)$$

So experimentally:

$$\Delta\theta = \frac{2\alpha}{\epsilon}\left(1 - \frac{r_0}{\alpha}\right)\int_{R_0}^{\infty} \frac{1}{r^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2} dr - \pi \quad -(13)$$

$$= \frac{4}{3}\frac{r_0}{R_0}$$

The distance of closest approach is defined

3) by:

$$R_0 = \frac{\alpha}{1+\epsilon} \quad -(14)$$

For the hyperbola:

$$\alpha = a(\epsilon^2 - 1) \quad -(15)$$

From eq. (14):

$$R_0(1+\epsilon) = \alpha \quad -(16)$$

so:

$$\Delta\theta = \frac{2R_0(1+\epsilon)}{\epsilon} \int_{R_0}^{\infty} \left(1 - \frac{r_0}{R_0(1+\epsilon)}\right) \left(\frac{1}{r^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)\right)^{-1/2} dr$$

$$- \pi$$

$$= \frac{4}{3}\frac{r_0}{R_0} \quad -(17)$$

Alternatively:

$$\epsilon = \frac{\alpha}{R_0} - 1 \quad -(18)$$

and $\Delta\theta$ can be evaluated from eq. (10) as a function of α.