263(4): Failure of the Newtonian Theory in Whirlpool Galaxies

It is worth emphasizing as in this note that the Newtonian theory fails completely in whirlpool galaxies, where the plot of linear orbital velocity v against distance r reaches a plateau. Here r is the distance between an orbiting star and the centre of the galaxy. The orbit of a star reaches a hyperbolic spiral in the infinite r limit. As discussed in Note 262(2) this observation can be explained with the plane polar coordinates. If the position vector is:

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(1)$$

the velocity is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(2)$$

so:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad -(3)$$

where

$$\omega = \dot{\theta} = \frac{d\theta}{dt} \quad -(4)$$

is the spin connection. The acceleration is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(5)$$

In any planar orbit eq. (5) simplifies to:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r \quad -(6)$$

Here

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \omega\frac{dr}{d\theta} \quad -(7)$$

The total angular momentum of the system is a constant of motion:

$$L = mr^2\omega \quad -(8)$$

so the spin connection is defined by:

$$\omega = \frac{L}{mr^2} \quad -(9)$$

The orbital velocity from fundamental plane polar geometry is:

$$v^2 = \left(\frac{L}{mr}\right)^2 + \left(\frac{L}{mr^2}\left(\frac{dr}{d\theta}\right)\right)^2 \quad -(10)$$

Note carefully that this result has been obtained without any assumption about a force law. It is a very powerful result of pure geometry, an example of Cartan geometry.

In a whirlpool galaxy it is observed that:

$$v \xrightarrow[r \to \infty]{} v_\infty \quad -(11)$$

for a star orbiting the central mass M. Here v_∞ = constant. From eq. (10):

$$v \xrightarrow[r \to \infty]{} \frac{L}{mr^2}\frac{dr}{d\theta} = v_\infty \quad -(12)$$

So

$$\frac{d\theta}{dr} \xrightarrow[r \to \infty]{} \left(\frac{L}{mv_\infty}\right)\frac{1}{r^2} \quad -(13)$$

and

$$\theta \xrightarrow[r \to \infty]{} \left(\frac{L}{mv_\infty}\right)\int \frac{dr}{r^2} \quad -(14)$$

So

$$\theta \xrightarrow[r \to \infty]{} -\left(\frac{L}{m v_\infty}\right)\frac{1}{r} \quad -(15)$$

This is a hyperbolic spiral. It is observed experimentally that the orbit of a star in the r to infinity limit is a hyperbolic spiral.

The experimental result has been explained from the pure geometry of the plane polar coordinates. The concept of force has not been used. This means that geometry is more fundamental than force. The stars travel on an orbit defined by geometry and not by force. This is a concept of general relativity.

Force is defined from the geometry using the equation:

$$\underline{F} = m\,\underline{a} \quad -(16)$$

For any orbit:

$$\underline{F} = m\left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r \quad -(17)$$

Eq. (17) can be rewritten as:

$$\underline{F}(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \quad -(18)$$

In the seventeenth century the force was defined as:

4)

$$\underline{F} = m\underline{a} = m\ddot{r}\underline{e}_r \quad -(19)$$

so a part of eq. (17) was missing. This is the centrifugal force:

$$\underset{\sim}{F}_c = -mr\dot{\theta}^2 \underline{e}_r \quad -(20)$$

$$= -\frac{L^2}{mr^3}\underline{e}_r$$

The concept of orbit is best understood using:

$$m\ddot{r}\underline{e}_r = \underset{\sim}{F}(r) + \frac{L^2}{mr^3}\underline{e}_r \quad -(21)$$

i.e

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3} \quad -(22)$$

The attractive force $F(r)$ is balanced by a repulsive centrifugal force.

From eq. (15) the force given by eq. (18) is

$$\underset{\sim}{F}(r) = -\frac{L^2}{mr^3}\underline{e}_r \quad -(23)$$

So

$$m\ddot{r} = -\frac{L^2}{mr^3} + \frac{L^2}{mr^3} = 0 \quad -(24)$$

in the r to infinity limit of a whirlpool galaxy. This means that

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = 0 \quad -(25)$$

and

$$\frac{dv}{dt} = \frac{d^2r}{dt^2} = 0. \quad -(26)$$

This implies self consistently that:

$$v_\infty = \frac{dr}{dt} = \text{constant} \quad -(27)$$

as observed experimentally.

The force law (23) was first derived in the eighteenth / seventeenth century by Roger Coates, long before whirlpool galaxies were known.

The orbit of planets in the solar system is an ellipse to an excellent approximation:

$$\frac{1}{r} = \frac{1}{\alpha}(1 + \epsilon \cos\theta) \quad -(28)$$

The force law from eqs. (18) and (28) is:

$$F = -\frac{mMG}{r^2} \quad -(29)$$

with

$$\alpha = \frac{L^2}{m^2 MG} \quad -(30)$$

so

$$m\ddot{r} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} \quad -(31)$$

and is again the result of geometry.

) Obviously the Newton and Coates orbits are completely different. The Newton theory fails completely because it was derived from an elliptical orbit, whereas the orbit of a star in a whirlpool galaxy is a Coates spiral.

From eq. (28):

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \quad -(32)$$

in which:

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(33)$$

So the Newtonian velocity from eq. (3) is:

$$v^2 = \omega^2 r^2 \left(\frac{2\alpha}{r} - \left(\frac{r}{\alpha}\right)^2 \left(1 - \epsilon^2\right)\right) \quad -(34)$$

$$= MG\left(\frac{2}{r} - \frac{1}{a}\right)$$

where

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(35)$$

, the semi major axis of the ellipse. Note that:

$$\frac{1}{a} = \frac{1-\epsilon^2}{\alpha} = \frac{1}{r}\left(1 + \epsilon\cos\theta\right)\left(1 - \epsilon^2\right) \quad -(36)$$

So

$$v^2\,(\text{Newton}) = \frac{MG}{r}\left(2 - \left(1-\epsilon^2\right)\left(1 + \epsilon\cos\theta\right)\right) \quad -(37)$$

It follows that:

$$v(\text{Newton}) \xrightarrow[r \to \infty]{} 0 \quad -(38)$$

because

$$-1 \leq \cos\theta \leq 1, \quad -(39)$$

and

$$0 < \epsilon < 1. \quad -(40)$$

So the Newton theory fails completely to describe the velocity curve of a whirlpool galaxy. Experimentally:

$$v \xrightarrow[r \to \infty]{} v_\infty = \text{constant} \quad -(41)$$

It is of key importance to note that the Einstein theory does no better. The Einstein theory was designed to describe a tiny precession in the elliptical orbit of a planet in the solar system, at a time when whirlpool galaxies were not known, and the Cartan torsion was not known.

Solar system precessions are tiny, and can be described by:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(41)$$

where

$$x \sim 1. \quad -(42)$$

Using eq. (41) in eq. (3) gives the result:

8) $$v^2 = \left(\frac{L}{mr^2}\right)\left(1 + \left(\frac{x\epsilon\sin(x\theta)}{1+\epsilon\cos(x\theta)}\right)^2\right) \quad -(43)$$

so $$v\,(\text{Einstein}) \xrightarrow[r\to\infty]{} 0 \quad -(44)$$

which is completely wrong, not just wrong. Eq. (44) follows from:

$$-1 \leq \sin(x\theta) \leq 1 \quad -(45)$$
$$-1 \leq \cos(x\theta) \leq 1$$
$$0 < \epsilon < 1.$$

The right result is obviously given by the force law (23) of the Cotes spiral. This is completely different from the Einstein force law:

$$\underset{\sim}{F}(r) = \left(-\frac{mMG}{r^2} - \frac{3GML^2}{mc^2r^4}\right)\underset{\sim}{e}_r \quad -(46)$$

and completely different from the Newtonian:

$$\underset{\sim}{F}(r) = -\frac{mMG}{r^2}\underset{\sim}{e}_r \quad -(47)$$

yet eq. (23) is given by geometry, which is the path of the hyperbolic spiral.