# 259(5): Derivation of the Schroedinger Equation from the Beltrami Equation for Momentum.

Assume that the linear momentum $\underline{p}$ obeys the Beltrami equation:

$$\underline{\nabla} \times \underline{p} = \kappa \underline{p} \quad -(1)$$

which implies

$$(\nabla^2 + \kappa^2)\underline{p} = 0 \quad -(2)$$

because

$$\underline{\nabla} \cdot \underline{p} = 0 \quad -(3)$$

If $\underline{p}$ is a momentum in a classical straight line then:

$$\kappa = 0. \quad -(4)$$

In general however $\underline{p}$ has intricate solutions.

Now quantize eq. (2):

$$\underline{p}\,\phi = -i\hbar\,\underline{\nabla}\phi \quad -(5)$$

so

$$(\nabla^2 + \kappa^2)\underline{\nabla}\phi = 0. \quad -(6)$$

Use:

$$\nabla^2 \underline{\nabla}\phi = \underline{\nabla}\nabla^2\phi \quad -(7)$$

and

$$\underline{\nabla}(\kappa^2\phi) = \kappa^2\underline{\nabla}\phi \quad -(8)$$

Eq. (8) assumes that:

$$\underline{\nabla} \kappa^2 = 0 \quad -(9)$$

Eqs. (6) to (8) give:

$$\underline{\nabla}\left(\nabla^2 \phi + \kappa^2 \phi\right) = \underline{0} \quad -(10)$$

A possible solution is:

$$\nabla^2 \phi + \kappa^2 \phi = 0 \quad -(11)$$

which is the Helmholtz equation for the scalar $\phi$, the wavefunction of quantum mechanics.

The Schroedinger equation for a free particle is obtained by applying eq. (5) to:

$$E = \frac{p^2}{2m} \quad -(12)$$

so

$$-\frac{\hbar^2}{2m} \nabla^2 \phi = E \phi \quad -(13)$$

and

$$\left(\nabla^2 + \frac{2Em}{\hbar^2}\right) \phi = 0 \quad -(14)$$

Eqs. (11) and (14) are the same if:

$$\kappa^2 = \frac{2Em}{\hbar^2} \quad -(15)$$

QED

3) Using:

$$\underline{p} = \hbar \kappa \quad - (16)$$

then

$$p^2 = 2Em \quad - (17)$$

which is eq. (12), QED.

Therefore the free particle Schroedinger equation is:

$$\underline{\nabla} \times \underline{p} = \left( \frac{2Em}{\hbar^2} \right)^{1/2} \underline{p} \quad - (18)$$

with

$$\underline{p} \psi = -i \hbar \underline{\nabla} \psi \quad - (19)$$

The origin of the Schroedinger equation is a Beltrami equation for $\underline{p}$.

It is well known that eq. (1) has intricate solutions of the type animated in UFT 258, so the nature of spacetime is such that linear momentum is not "linear" in general, it has a rich variety of solutions