259(3): Higher Topology Electrodynamics and the Beltrami Equation.

In a higher topology electrodynamics such as ECE there is a richly structured Beltrami electrodynamics inherent through the entire development. This theory gives rise to a vast amount of new information.

Consider for example the magnetic flux density:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(1)$$

In general:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_{(1)} \times \underline{A}^{(1)} - \underline{\omega}^a{}_{(2)} \times \underline{A}^{(2)} - \underline{\omega}^a{}_{(3)} \times \underline{A}^{(3)} \quad -(2)$$

Assume that:

$$\underline{\omega}^a{}_b = \epsilon^a{}_{bc}\, \underline{\omega}^c \quad -(3)$$

so that the tensor $\underline{\omega}^a{}_b$ in the internal space is dual to the vector $\underline{\omega}^c$. Then:

$$\underline{\omega}^{(1)}{}_{(2)} = \epsilon^{(1)}{}_{(2)(3)}\, \underline{\omega}^{(3)} = \underline{\omega}^{(3)} \quad -(4)$$

$$\underline{\omega}^{(1)}{}_{(3)} = \epsilon^{(1)}{}_{(3)(2)}\, \underline{\omega}^{(2)} = -\underline{\omega}^{(2)} \quad -(5)$$

So:

$$\underline{B}^{(1)} = \underline{\nabla} \times \underline{A}^{(1)} - \underline{\omega}^{(3)} \times \underline{A}^{(2)} + \underline{\omega}^{(2)} \times \underline{A}^{(3)} \quad -(6)$$

et cyclicum

2) In the absence of a magnetic monopole, the Cartan identity is:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^b = 0 \quad -(7)$$

which implies:

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b = \underline{A}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(8)$$

Eq. (3) is a possible solution of eq. (8), i.e. eq. (3) is a possible solution of the Cartan identity of higher topology geometry. This gives a rigorous geometrical justification for O(3) electrodynamics.

The Cartan identity (7) is itself a Beltrami equation:

$$\underline{\nabla} \times (\underline{\omega}^a{}_b \times \underline{A}^b) = \kappa\, \underline{\omega}^a{}_b \times \underline{A}^b \quad -(9)$$

From eqs. (3) and (9):

$$\boxed{\underline{\nabla} \times (\underline{A}^c \times \underline{A}^b) = \kappa\, \underline{A}^c \times \underline{A}^b} \quad -(10)$$

In the complex circular basis:

$$\underline{A}^{(1)} \times \underline{A}^{(2)} = i A^{(0)} \underline{A}^{(3)*} \quad -(11)$$

et cyclicum

i.e.

$$\underline{A}^{(2)} \times \underline{A}^{(3)} = i A^{(0)} \underline{A}^{(1)*} \quad -(12)$$

$$\underline{A}^{(3)} \times \underline{A}^{(1)} = i A^{(0)} \underline{A}^{(2)*} \quad -(13)$$

3) From eqs. (10) to (13):

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(14)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} \quad -(15)$$

$$\underline{\nabla} \times \underline{A}^{(3)} = \kappa \underline{A}^{(3)} \quad -(16)$$

QED. These results were derived in another way by considering plane waves in the series of ECE papers. They can be derived self consistently in another way by using the Gauss Law:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(17)$$

which implies the Beltrami equation:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \quad -(18)$$

From eqs. (1) and (18):

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a = \kappa \left(\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \right) \quad -(19)$$

so:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}^a) - \underline{\nabla} \times (\underline{\omega}^a{}_b \times \underline{A}^b) \quad -(20)$$
$$= \kappa (\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b).$$

Using eq. (9) gives:

4)

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}^a) = \kappa \underline{\nabla} \times \underline{A}^a \quad -(21)$$

which gives (14) to (16) in another way, Q.E.D.

Therefore in higher topology electrodynamics the vector potential always obeys a Beltrami equation:

$$\boxed{\underline{\nabla} \times \underline{A}^a = \kappa \underline{A}^a} \quad -(22)$$

so

$$\boxed{\underline{\nabla} \cdot \underline{A}^a = 0} \quad -(23)$$

The equation of charge current conservation is:

$$\frac{1}{c^2} \frac{\partial \phi^a}{\partial t} + \underline{\nabla} \cdot \underline{A}^a = 0 \quad -(24)$$

so:

$$\boxed{\frac{\partial \phi^a}{\partial t} = 0} \quad -(25)$$

From eqs. (8) and (22):

$$\boxed{\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa \underline{\omega}^a{}_b} \quad -(26)$$

so the spin connection vector also obeys a Beltrami equation

5) It follows that:

$$\underline{\omega}^{(3)} = \frac{1}{2}\frac{\kappa}{A^{(0)}}\underline{A}^{(3)} \quad -(27)$$

and

$$\underline{\omega}^{(2)} = \frac{1}{2}\frac{\kappa}{A^{(0)}}\underline{A}^{(2)} \quad -(28)$$

produces the well known results of O(3) electrodynamics from Cartan geometry:

$$\underline{B}^{(1)*} = \underline{\nabla}\times\underline{A}^{(1)*} - i\frac{\kappa}{A^{(0)}}\underline{A}^{(2)}\times\underline{A}^{(3)} \quad -(29)$$

$$\underline{B}^{(2)*} = \underline{\nabla}\times\underline{A}^{(2)*} - \frac{i\kappa}{A^{(0)}}\underline{A}^{(3)}\times\underline{A}^{(1)} \quad -(30)$$

$$\underline{B}^{(3)*} = \underline{\nabla}\times\underline{A}^{(3)*} - \frac{i\kappa}{A^{(0)}}\underline{A}^{(1)}\times\underline{A}^{(2)} \quad -(31)$$

These are all relations between very richly structured Beltrami fields.

The original plane waves of O(3) electrodynamics are the simplest examples. It follows from eqs. (11) - (13), (22), and (29) - (31) that:

$$\underline{B}^{(1)*} = \underline{\nabla} \times \underline{A}^{(1)*} + \kappa \underline{A}^{(1)*} = 2\kappa \underline{A}^{(1)*} \quad -(32)$$

$$\underline{B}^{(2)*} = \underline{\nabla} \times \underline{A}^{(2)*} + \kappa \underline{A}^{(2)*} = 2\kappa \underline{A}^{(2)*} \quad -(33)$$

$$\underline{B}^{(3)*} = \underline{\nabla} \times \underline{A}^{(3)*} + \kappa \underline{A}^{(3)*} = 2\kappa \underline{A}^{(3)*} \quad -(34)$$

in general

Therefore <u>U(1) gauge invariance</u> is <u>no longer valid</u> in <u>higher topology electrodynamics</u>. This means the existence of <u>photon mass</u> and the $\underline{B}^{(3)}$ field.

From eqs. (22) and (32) - (34):

$$\underline{A}^{(1)} = \frac{1}{2\kappa^2} \underline{\nabla} \times \underline{B}^{(1)} \quad -(35)$$

$$\underline{A}^{(2)} = \frac{1}{2\kappa^2} \underline{\nabla} \times \underline{B}^{(2)} \quad -(36)$$

For the simple plane wave solution of the Beltrami equation:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)} \quad -(37)$$

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i(\omega t - \kappa Z)} \quad -(38)$$

$$\underline{A}^{(3)} = A^{(0)} \underline{k} \quad -(39)$$

7)

so:

$$\underline{\nabla} \times \underline{A}^{(3)} = 0 \, \underline{A}^{(3)}, \quad -(40)$$

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(41)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} \quad -(42)$$

so in this case:

$$\boxed{\underline{B}^{(3)*} = \underline{B}^{(3)} = 2\kappa \underline{A}^{(3)}} \quad -(43)$$

where κ is defined by eqs. (41) and (42).

From the simple eq. (39):

$$\underline{\nabla} \times \underline{A}^{(3)} = 0 \underline{A}^{(3)} \quad -(44)$$

It follows that:

$$\underline{\nabla} \times \underline{B}^{(1)} = \kappa \underline{B}^{(1)} \quad -(45)$$

$$\underline{\nabla} \times \underline{B}^{(2)} = \kappa \underline{B}^{(2)} \quad -(46)$$

$$\underline{\nabla} \times \underline{B}^{(3)} = 0 \, \underline{B}^{(3)} \quad -(47)$$

Finally consider the equations:

$$\underline{A}^{(1)*} = \underline{A}^{(2)} = -\frac{i}{A^{(0)}} \underline{A}^{(2)} \times \underline{A}^{(3)} \quad -(48)$$

and

$$\underline{A}^{(3)} = \frac{1}{\kappa} \underline{B}^{(3)} \quad -(49)$$

From eqs. (48) and (49):

$$\underline{A}^{(2)} = -\frac{i}{\kappa A^{(0)}} \underline{A}^{(2)} \times \underline{B}^{(3)} = -i \underline{r}^{(2)} \times \underline{B}^{(3)} \quad -(50)$$

where

$$\underset{\sim}{r}^{(2)} = \frac{1}{\kappa} \underline{e}^{(2)} \quad -(51)$$

with

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) \quad -(52)$$

and

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} \quad -(53)$$

In the static limit:

$$\omega t - \kappa Z \rightarrow 0 \quad -(54)$$

assuming that the photon mass is negligible for all practical purposes, but identically non zero. More accurately:

$$\left(\frac{\omega}{c}\right)^2 = \kappa^2 + \left(\frac{mc}{\hbar}\right)^2 \quad -(55)$$

where the wavenumber of the photon at rest is the rest wavenumber:

$$\kappa_0 = \frac{mc}{\hbar} \quad -(56)$$

Assuming eq. (54), eq. (50) becomes:

4)

$$\underline{A}^{(2)} = -i\underline{r}^{(2)} \times \underline{B}^{(3)} \quad -(57)$$

where

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right), \quad -(58)$$

$$\underline{r}^{(2)} = \frac{1}{\kappa\sqrt{2}}\left(\underline{i} + i\underline{j}\right), \quad -(59)$$

$$\underline{B}^{(3)} = B^{(0)}\underline{k} = B_Z\underline{k} \quad -(60)$$

in which:

$$\underline{\nabla} \times \underline{A}^{(2)} \rightarrow \kappa \underline{A}^{(2)} \quad -(61)$$

$$\underline{\nabla} \times \underline{B}^{(3)} = 0 \underline{B}^{(3)} \quad -(62)$$

Eq. (57) is the correct equation of the static magnetic field in higher topology electrodynamics. In the U(1) or Heaviside electrodynamics:

$$\underline{A} = \frac{1}{2}\underline{B} \times \underline{r} = \frac{B_Z}{2}\left(-Y\underline{i} + X\underline{j}\right) \quad -(63)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} = B_Z\underline{k} \quad -(64)$$

So in U(1) electrodynamics $\underline{A}$ is not a correct Beltrami potential because $\underline{\nabla} \times \underline{A}$ is not // $\underline{A}$.