259(4): Beltrami, Helmholtz and Schroedinger Equations

In ECE electrodynamics the magnetic field is

$$\underline{B}^{(3)} = \underline{\nabla} \times \underline{A}^{(3)} - i\frac{\kappa}{A^{(0)}} \underline{A}^{(1)} \times \underline{A}^{(2)} \quad -(1)$$

in general in which:

$$p = \hbar \kappa = eA^{(0)} \quad -(2)$$

The potentials are related by:

$$\underline{A}^{(1)} \times \underline{A}^{(2)} = i A^{(0)} \underline{A}^{(3)*} \quad -(3)$$

et cyclicum

and are Beltrami functions:

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(4)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} \quad -(5)$$

with:

$$\underline{A}^{(1)} = \underline{A}^{(2)*} \quad -(6)$$

These structures follow from Cartan geometry and are true in general.

For plane waves:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)} \quad -(7)$$

2)

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i(\omega t - kz)} \quad - (8)$$

$$\underline{A}^{(3)} = A^{(0)} \underline{k} \quad - (9)$$

So:

$$\underline{B}^{(3)} = -\frac{ik}{A^{(0)}} \underline{A}^{(1)} \times \underline{A}^{(2)}, \quad - (10)$$

which is the fundamental $\underline{B}^{(3)}$ field of propagating electromagnetic radiation.

It can be written as:

$$\underline{B} = -i\frac{e}{\hbar} \underline{A} \times \underline{A}^{*} \quad - (11)$$

$$= B^{(0)} \underline{k} = B_Z \underline{k}$$

and in this format can be used as the definition of a static magnetic field. This is important for the subject of magnetostatics, and the development of the fermion equation with:

$$\underline{A} = \frac{1}{2} \underline{B} \times \underline{r} \quad - (12)$$

Eq. (11) also gives the transition from classical to quantum mechanics.

3)

In ECE electrodynamics, $\underline{A}$ must always be a Beltrami field. As shown in recent work this is the direct result of the Cartan identity. So it is necessary to solve:

$$\underline{B} = -i \frac{e}{\hbar} \underline{A} \times \underline{A}^* \quad -(13)$$

$$\underline{A} = \frac{1}{2} \underline{B} \times \underline{r} \quad -(14)$$

$$\underline{\nabla} \times \underline{A} = \kappa \underline{A} \quad -(15)$$

This can be done by using the principles of general relativity, so that the electromagnetic field is a rotating and translating frame of reference. The position vector is therefore:

$$\underline{r} = \frac{r^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(16)$$

$$\underline{r}^* = \frac{r^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} \quad -(17)$$

where:

$$\underline{r} = \underline{r}^{(1)}, \quad \underline{r}^* = \underline{r}^{(2)} \quad -(18)$$

$$\underline{r}^{(1)} \times \underline{r}^{(2)} = i r^{(0)} \underline{r}^{(3)*} \quad -(19)$$

et cyclicum

It follows that:

$$\underline{\nabla} \times \underline{r}^{(1)} = \kappa \, \underline{r}^{(1)} \quad -(20)$$

$$\underline{\nabla} \times \underline{r}^{(2)} = \kappa \, \underline{r}^{(2)} \quad -(21)$$

$$\underline{\nabla} \times \underline{r}^{(3)} = 0 \, \underline{r}^{(3)} \quad -(22)$$

The results (16) to (22) for plane wave can be generalized to any Beltrami solutions. It follows that spacetime itself has a Beltrami structure.

From eqs. (14) and (16):

$$\underline{A} = \underline{A}^{(1)} = \frac{B^{(0)} r^{(0)}}{2\sqrt{2}} \left(i\underline{i} + \underline{j} \right) e^{i\phi} \quad -(23)$$

$$= \frac{A^{(0)}}{\sqrt{2}} \left(i\underline{i} + \underline{j} \right) e^{i\phi}$$

where

$$A^{(0)} = \frac{1}{2} B^{(0)} r^{(0)} \quad -(24)$$

From eq. (23):

$$\underline{\nabla} \times \underline{A} = \kappa \underline{A} \quad -(25)$$

QED. Therefore it is always possible to write the vector potential in the form of

5) eq. (12) provided that spacetime itself has a Beltrami structure. This conclusion ties together several branches of physics because eq. (12) is used to produce the Landé factor, ESR, NMR and so on from the Dirac equation, which becomes the fermion equation of ECE physics.

In ECE physics the tetrad postulate of Cartan geometry gives:

$$(\square + \kappa_0^2) \underline{A} = \underline{0} \quad -(26)$$

under all conditions. Eq. (26), as always in ECE physics, is the result of geometry. The fermion equation, or chiral Dirac equation, is a factorization of eq. (26) as shown in previous work. The wave-number κ_0 is in general the result of Cartan geometry as shown many times in previous work.

For ease of reference eq. (26) is derived as follows from Cartan geometry.

Consider the tetrad postulate:

$$D_\mu q^a_{\ \nu} = \partial_\mu q^a_{\ \nu} + \omega^a_{\ \mu b} q^b_{\ \nu} - \Gamma^\lambda_{\ \mu\nu} q^a_{\ \lambda} = 0 \quad -(27)$$

$$= \partial_\mu q^a_{\ \nu} + \omega^a_{\ \mu\nu} - \Gamma^a_{\ \mu\nu}$$

) It follows that:

$$\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(28)$$

and

$$\Box q^a_\nu = \partial^\mu \partial_\mu q^a_\nu = \partial^\mu \left(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \right) \quad -(29)$$

$$= q^\nu_a \, \partial^\mu \left(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \right) q^a_\nu$$

using

$$q^\nu_a \, q^a_\nu = 1 \, . \quad -(30)$$

Define:

$$\boxed{\kappa_0^2 := q^\nu_a \, \partial^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} \right)} \quad -(31)$$

to obtain

$$\left(\Box + \kappa_0^2 \right) q^a_\nu = 0 \quad -(32)$$

Note carefully that eq. (32) is the tetrad postulate of Cartan geometry.

Now use the ECE hypothesis:

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(33)$$

to find that:

$$\left(\Box + \kappa_0^2 \right) A^a_\mu = 0 \quad -(34)$$

Finally use:

7)

$$A^a_\mu = \left(A^a_0, -\underline{A}^a \right) \quad -(35)$$

so that for each a:

$$\left(\Box + \kappa_0^2 \right) A_0 = 0 \quad -(36)$$

and

$$\left(\Box + \kappa_0^2 \right) \underline{A} = 0 \quad -(37)$$

which gives (26) QED

The d'Alembertian is defined by:

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad -(38)$$

From eq. (25):

$$\underline{\nabla} \times \left(\underline{\nabla} \times \underline{A} \right) = \kappa \, \underline{\nabla} \times \underline{A} = \kappa^2 \underline{A} \quad -(39)$$

and

$$\underline{\nabla} \cdot \underline{A} = 0 \quad -(40)$$

because:

$$\underline{A} = \frac{1}{\kappa} \underline{\nabla} \times \underline{A} \quad -(41)$$

and

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A} := 0 \quad -(42)$$

From vector analysis:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla} (\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A} \quad -(43)$$

So from eqs. (39) to (43):

$$\boxed{(\nabla^2 + \kappa^2) \underline{A} = 0} \quad -(44)$$

which is the Helmholtz wave equation. In ECE electrodynamics this is true for each a:

$$(\nabla^2 + \kappa^2) \underline{A}^a = 0 \quad -(45)$$

The Helmholtz wave equation is the result of the Beltrami equation.

From eq. (37):

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \kappa_0^2 \right) \underline{A} = \underline{0} \quad -(46)$$

so

$$\boxed{\frac{1}{c^2} \frac{\partial^2 \underline{A}}{\partial t^2} + (\kappa_0^2 + \kappa^2) \underline{A} = \underline{0}} \quad -(47)$$

This is the equation for the time dependence of $\underline{A}$.

The Helmholtz and Beltrami equations are equations for the space dependence of $\underline{A}$. Eq. (47) is satisfied by:

$$\underline{A} = \underline{A}_0 e^{i\omega t} \quad -(48)$$

where

$$\frac{\omega^2}{c^2} = \kappa^2 + \kappa_0^2 \quad -(49)$$

Eq. (49) is a generalization of the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(50)$$

using:

$$E = \hbar\omega, \quad p = \hbar\kappa, \quad -(51)$$

$$\boxed{\kappa_0^2 = \left(\frac{mc}{\hbar}\right)^2 = q^{\nu}_{a} \partial^{\mu} \left(\omega^{a}_{\mu\nu} - \Gamma^{a}_{\mu\nu}\right)}$$

So mass m in ECE theory is defined by geometry.

The general solution of eq. (34) is therefore:

$$A^a_\mu = A^a_\mu(0) \exp\left(i(\omega t - \kappa Z)\right) \quad -(52)$$

where

$$\omega^2 = c^2(\kappa^2 + \kappa_0^2) \quad -(53)$$

It follows that there exists:

$$(\Box + \kappa_0^2)\phi^a = 0 \quad - (54)$$

and

$$(\nabla^2 + \kappa^2)\phi^a = 0 \quad - (55)$$

where ϕ^a is the scalar potential in ECE physics.

For each a:

$$\nabla^2 \phi = -\kappa^2 \phi . \quad - (56)$$

Now write:

$$\kappa_0 = \frac{mc}{\hbar} \quad - (57)$$

where m is a mass. For this mass the relativistic wave equation for each a is:

$$(\Box + \kappa_0^2)\phi = 0 \quad - (58)$$

which is the quantized form of:

$$\begin{aligned} E^2 &= c^2 p^2 + m^2 c^4 \\ &= c^2 p^2 + \kappa_0^2 \hbar^2 c^2 \end{aligned} \quad - (59)$$

Eq. (59) is:

$$E = \gamma m c^2 \quad - (60)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (61)$$

) and $$\underline{p} = \gamma m \underline{v} \quad -(62)$$

Define the relativistic kinetic energy as:

$$T = E - mc^2 \quad -(63)$$

and it follows that:

$$\begin{aligned} T &= (\gamma - 1) mc^2 \\ &= \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right) mc^2 \quad -(64) \\ &\sim \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) mc^2 \\ &= \frac{1}{2} m v^2 \end{aligned}$$

which is the non-relativistic limit of the kinetic energy:

$$T = \frac{p^2}{2m} \quad -(65)$$

Using:

$$T = i\hbar \frac{\partial}{\partial t} \quad -(66)$$

$$\underline{p} = -i\hbar \underline{\nabla} \quad -(67)$$

eq. (65) quantizes to the free particle Schroedinger equation

(a)

$$-\frac{\hbar^2}{2m}\nabla^2\phi = T\phi \quad -(68)$$

i.e.

$$\nabla^2\phi = -\left(\frac{2m}{\hbar^2}T\right)\phi \quad -(69)$$

It is seen that eqs. (56) and (69) have the same structure. The Helmholtz and free particle Schroedinger equations have the same structure and same solution.

It follows that the free particle Schroedinger equation is a Beltrami equation but with the vector potential $\underline{A}$ replaced by the scalar ϕ. The scalar potential plays the role of the wave function. It also follows in the non-relativistic limit that:

$$\nabla^2\underline{A} + \left(\frac{2mT}{\hbar^2}\right)\underline{A} = 0 \quad -(70)$$

so

$$\kappa^2 = \frac{2mT}{\hbar^2} \quad -(71)$$

13) Eq. (69) can be written as:

$$\nabla^2 \phi + \kappa^2 \phi = 0 \quad -(72)$$

which is an Euler-Bernoulli equation without a driving term or damping term

In the presence of potential energy V eq (68) becomes:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \phi = E \phi \quad -(73)$$

$$:= H \phi$$

where H is the hamiltonian and E the total energy:

$$E = T + V \quad -(74)$$

Eq. (73) is:

$$(\nabla^2 + \kappa^2) \phi = \frac{2mV}{\hbar^2} \phi \quad -(75)$$

which is similar to an Euler Bernoulli equation with a driving term on the right hand side. However eq. (75) is an eigen equation but the well known resonant solutions. Eq. (75) may be written as:

$$(\nabla^2 + \kappa_1^2)\,\phi = 0 \quad - (76)$$

where

$$\kappa_1^2 = \frac{2m}{\hbar^2}(E - V) \quad - (77)$$

and in UFT 226 ff. was used in the theory of low energy nuclear reactors.

As described by Marion and Thornton, eq. (76) is a linear oscillator equation. It can be used to describe the structure of the atom and the nucleus. It can be transformed into an Euler Bernoulli equation as follows:

$$(\nabla^2 + \kappa_1^2)\,\phi = A \cos(\kappa_2 Z) \quad - (78)$$

where the right hand side represents a vacuum potential.

Eq. (78) is precisely the kind of structure obtained from the ECE Coulomb law.