: Beltrami Fields and the Proca Equation

In ECE physics the Proca equation may be written as:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad - (1)$$

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = 0 \quad - (2)$$

$$\underline{\nabla} \cdot \underline{E}^a = \frac{\rho^a}{\epsilon_0} \quad - (3)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2} \frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \quad - (4)$$

in the usual notation of these notes. Define the four current and four potential respectively by:

$$J^\mu = \left(c\rho, \underline{J} \right) \quad - (5)$$

$$A^\mu = \left(\frac{\phi}{c}, \underline{A} \right) \quad - (6)$$

Here the Proca theory on the ECE level asserts that:

$$J^\mu = -\epsilon_0 \left(\frac{mc}{\hbar} \right)^2 A^\mu \quad - (7)$$

where m is the photon mass.

Therefore:

$$\rho^a = -\epsilon_0 c^2 \left(\frac{mc}{\hbar} \right)^2 \phi^a \quad - (8)$$

$$\underline{J}^a = -\epsilon_0 \left(\frac{mc}{\hbar} \right)^2 \underline{A}^a \quad - (9)$$

On the U(1) level the Proca equation is:

2.)

$$\partial_\mu F^{\mu\nu} = \tilde{J} / \epsilon_0 \quad - (10)$$

$$= -\left(\frac{mc}{\hbar}\right)^2 \tilde{A}$$

Therefore:

$$\partial_\nu \partial_\mu F^{\mu\nu} = \frac{1}{\epsilon_0} \partial_\nu \tilde{J}$$

$$= -\left(\frac{mc}{\hbar}\right)^2 \partial_\nu \tilde{A} \quad - (11)$$

$$= 0$$

So:

$$\boxed{\partial_\mu J^\mu = \partial_\mu A^\mu = 0} \quad - (12)$$

Therefore the Proca theory condiðe the conservation of charge current density and the Lorenz condition.

In vector notation eq. (11) is:

$$\frac{1}{c}\frac{d}{dt}\left(\underline{\nabla} \cdot \underline{E}\right) = \frac{1}{c\epsilon_0}\frac{d\rho}{dt} = 0 \quad - (13)$$

and

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{d}{dt}\underline{\nabla} \cdot \underline{E} = \mu_0 \underline{\nabla} \cdot \underline{J} \quad - (14)$$

Now we

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{B} = 0 \quad - (15)$$

and
$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad - (16)$$

to find that

$$-\frac{1}{c^2 \epsilon_0} \frac{\partial \rho}{\partial t} = \mu_0 \underline{\nabla} \cdot \underline{J} \qquad - (17)$$

with

$$\mu_0 \epsilon_0 = \frac{1}{c^2} \qquad - (18)$$

so:

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0} \qquad - (19)$$

This is the conservation of charge current density:

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0. \qquad - (20)$$

In the Proca formalism eq. (20) implies the Lorenz condition:

$$\partial_\mu A^\mu = \frac{1}{c^2} \frac{\partial \phi}{\partial t} + \underline{\nabla} \cdot \underline{A} = 0 \qquad -(21)$$

Therefore the Proca equation is not $u(1)$ gauge invariant because:

$$J^\mu = -\epsilon_0 \left(\frac{mc}{\hbar}\right)^2 A^\mu \qquad -(22)$$

and the idea of $u(1)$ gauge invariance is

4) What physical quantities are unchanged under:

$$A^\mu \to A^\mu + \partial^\mu \phi \quad -(23)$$

where ϕ is an arbitrary function. Eq. (22) shows that if there is photon mass, J^μ changes under the transformation (23)

Photon mass is <u>incompatible</u> with <u>U(1)</u> <u>gauge</u> <u>invariance</u>.

The usual theory of U(1) gauge invariance means that:

$$F^{\mu\nu} \to F^{\mu\nu} + (\partial^\mu \partial^\nu - \partial^\nu \partial^\mu) \chi$$

$$= F^{\mu\nu}. \quad -(24)$$

The Proca wave equation on the U(1) level is obtained from:

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = -\left(\frac{mc}{\ell}\right)^2 \tilde{A}^\nu \quad -(25)$$

i.e

$$\Box A^\nu - \partial_\mu \partial^\nu A^\mu + \left(\frac{mc}{\ell}\right)^2 A^\nu = 0. \quad -(26)$$

Now use:

$$\partial_\mu \partial^\nu A^\mu = \partial^\nu \partial_\mu A^\mu = 0 \quad -(27)$$

from eqv. (12), so the Proca wave equation is:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) \tilde{A} = 0 \quad - (28)$$

In ECE physics the Proca equation (28) is obtained from the tetrad postulate and the ECE hypothesis, and is:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) A_\mu^a = 0 \quad - (29)$$

The Proca field equations are obtained from the Cartan and Evans identities.

The conservation of charge current density at ECE level is:

$$\boxed{\frac{\partial \rho^a}{\partial t} + \underline{\nabla} \cdot \underline{J}^a = 0} \quad - (30)$$

and is obtained from eqs. (3) and (4):

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{B}^a - \frac{1}{c^2} \frac{\partial}{\partial t} \underline{\nabla} \cdot \underline{E}^a = \mu_0 \underline{\nabla} \cdot \underline{J}^a \quad - (31)$$

i.e

$$-\frac{1}{c^2 \epsilon_0} \frac{\partial \rho^a}{\partial t} = \mu_0 \underline{\nabla} \cdot \underline{J}^a \quad - (32)$$

which is eq. (30), QED.

Therefore conservation of charge-current density

') is the result of _geometry_, as is _photon mass._

In ECE physics the electric charge density is:

$$\rho^a = \epsilon_0 \left(\underline{\omega}^a{}_b \cdot \underline{E}^b - c\underline{A}^b \cdot \underline{R}^a{}_b \,(orb) \right) \quad -(33)$$

and

$$\underline{J}^a = \frac{1}{\mu_0} \left(\underline{\omega}^a{}_b \times \underline{B}^b + \frac{\omega_0}{c} \underline{E}^b \right. \quad -(34)$$
$$\left. - \left(\underline{A}^b \times \underline{R}^a{}_b \,(spin) + A^b \cdot \underline{R}^a{}_b \,(orb) \right) \right)$$

Therefore it is possible to use eqs. $(30), (33)$ and (34) to obtain _new equations._

Also, in photon mass theory eqs. (8) and (9) can be applied with eqs $(30), (33)$ and (34).

__Illustration with Magnetostatics__

In this case the only thing present is a static magnetic field, so:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(35)$$
$$\underline{\nabla} \times \underline{B}^a = \mu_0 \underline{J}^a \quad -(36)$$

and

$$\underline{\nabla} \cdot \underline{J}^a = \underline{\nabla} \cdot \underline{\nabla} \times \underline{B}^a = 0 \quad -(37)$$

so

$$\partial \rho^a / \partial t = 0 \quad -(38)$$

) There is no electric field present and if it is assumed that the scalar potential is zero:

$$J^a = \frac{1}{\mu_0} \left(\underline{\omega}^a{}_b \times \underline{B}^b - \underline{A}^b \times \underline{R}^a{}_b \,(\text{spin}) \right) \quad -(39)$$

so it follows from eq. (37) that:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{B}^b = \underline{\nabla} \cdot \underline{A}^b \times \underline{R}^a{}_b \,(\text{spin}) \quad \longrightarrow (40)$$

In the absence of a magnetic monopole:

$$\underline{\omega}^a{}_b \cdot \underline{B}^b = \underline{A}^b \cdot \underline{R}^a{}_b \,(\text{spin}) \quad -(41)$$

Using vector analysis:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{B}^b = \underline{B}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{B}^b \quad -(42)$$

$$\underline{\nabla} \cdot \underline{A}^b \times \underline{R}^a{}_b \,(\text{spin}) = \underline{R}^a{}_b \,(\text{spin}) \cdot \underline{\nabla} \times \underline{A}^a \quad -(43)$$
$$-\underline{A}^b \cdot \underline{\nabla} \times \underline{R}^a{}_b \,(\text{spin})$$

From the space part of the Cartan identity:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^b = 0 \quad -(44)$$

in the absence of a magnetic monopole.

), and :

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b = \underline{A}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad - (45)$$

The magnetic flux density is:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad - (46)$$

The **Beltrami equation** is :

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a = \kappa \left(\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \right) \\ - (47)$$

Eq. (44) implies the Beltrami equation :

$$\underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{A}^b \right) = \kappa \, \underline{\omega}^a{}_b \times \underline{A}^b \quad - (48)$$

From eq. (47) :

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad - (49)$$

so :

$$\underline{\nabla} \times \left(\underline{\nabla} \times \underline{A}^a \right) - \underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{A}^b \right)$$
$$= \underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \qquad - (50)$$
$$= \kappa \left(\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \right)$$

using eq. (48) :

9)

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}^a) = k \, \underline{\nabla} \times \underline{A}^a \quad - (51)$$

i.e.

$$\boxed{\underline{\nabla} \times \underline{A}^a = k \underline{A}^a} \quad - (52)$$

Using eqs. (45) and (52):

$$\boxed{\underline{\nabla} \times \underline{\omega}^a{}_b = k \, \underline{\omega}^a{}_b} \quad - (53)$$

Fm eq. (42), (53) and (47):

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{B}^b = 0 \quad - (54)$$

so fm eq. (40):

$$\underline{\nabla} \cdot \underline{A}^b \times \underline{R}^a{}_b (\text{spin}) = 0 \quad - (55)$$

Fm eq. (43):

$$\underline{R}^a{}_b (\text{spin}) \cdot \underline{\nabla} \times \underline{A}^a = \underline{A}^b \cdot \underline{\nabla} \times \underline{R}^a{}_b (\text{spin})$$
$$- (56)$$

s. fm eq. (52):

$$\boxed{\underline{\nabla} \times \underline{R}^a{}_b (\text{spin}) = k \, \underline{R}^a{}_b (\text{spin})} - (57)$$

for magnetostatics.

In order to summarize these results, the following equations are generally true:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a$$

$$\underline{\nabla} \times \underline{A}^a = \kappa \underline{A}^a \qquad -(58)$$

$$\underline{\nabla} \times \underline{\omega}^a_b = \kappa \underline{\omega}^a_b$$

and for magnetostatics :

$$\underline{\nabla} \times \underline{R}^a_b (\text{spin}) = \kappa \underline{R}^a_b (\text{spin}) \quad -(59)$$

Also for magnetostatics:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a = \mu_0 \underline{J}^a \quad -(60).$$

So

$$\underline{\nabla} \times \underline{J}^a = \kappa \underline{J}^a \qquad -(61)$$

Therefore in photon mass theory :

$$\underline{\nabla} \times \underline{A}^a = \kappa \underline{A}^a \quad -(62)$$

and

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \underline{A}^a = 0 \quad -(63)$$

It follows from eq. (62) that :

$$\underline{\nabla} \cdot \underline{A}^a = 0 \qquad -(64)$$

therefore eq. (63) becomes :

$$\nabla^2 \underline{A}^a + \kappa^2 \underline{A}^a \cdots = 0 \quad - (65)$$

which is the Helmholtz equation. Eq. (63) is:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right) \underline{A}^a = \underline{0} \quad - (66).$$

so $\qquad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \kappa^2 + \left(\frac{mc}{\hbar} \right)^2 \right) \underline{A}^a = \underline{0} \quad - (67)$

Now we:

$$p = \hbar \kappa \quad - (68)$$

and

$$\frac{\partial^2}{\partial t^2} = - \frac{E^2}{\hbar^2} \quad - (69)$$

to find that eq. (67) is the Einstein energy equation:

$$E^2 = p^2 c^2 + m^2 c^4 \quad - (70)$$

for photon mass.

It is concluded that in ECE physics

$$\partial_\mu A^{a\mu} = 0 \quad - (71)$$

i.e.

$$\frac{1}{c^2} \frac{\partial \phi^a}{\partial t} + \underline{\nabla} \cdot \underline{A}^a = 0 \quad - (72)$$

12) and

$$\underline{\nabla} \cdot \underline{A}^a = 0 \quad - (73)$$

$$\frac{\partial \phi^a}{\partial t} = 0 \quad - (74)$$

So:

Also in ECE physics:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b = 0 \quad - (75)$$

and the spin connection vector is a Beltrami
vector.

The magnetic field is defined by:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^a \quad - (76)$$

So:

$$\underline{\nabla} \cdot \underline{B}^a = - \underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^a = 0 \quad - (77)$$

because:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^a = 0 \quad - (78)$$

By vector analysis:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^a = \underline{A}^a \cdot \underline{\nabla} \times \underline{\omega}^a{}_b$$
$$- \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b$$
$$= 0 \quad - (79)$$

because:

$$\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa \, \underline{\omega}^a{}_b. \quad - (80)$$

and
$$\underline{\nabla} \times \underline{A}^b = \kappa \, \underline{A}^b. \quad - (81)$$

i.e because:

$$\underline{\nabla} \cdot \underline{A}^a = 0 \quad - (82)$$

and
$$\underline{\nabla} \cdot \underline{\omega}^a{}_b = 0 \quad - (83)$$

In the absence of a magnetic monopole eq. (79) also follows from the space part of the Cartan identity.

Therefore the entire analysis is rigorously self consistent.

Finally in photon mass theory eqns. (8) and (9) show that:

$$\frac{d\rho^a}{dt} = 0 \quad - (84)$$

and
$$\underline{\nabla} \cdot \underline{J}^a = 0 \quad - (85)$$

The charge density associated with photon mass is independent of time and the current density associated with photon mass has no

14) divergence.

The photon mass may be thought of as
an entity which generates a vacuum
current from a vacuum four potential; as
it equns (8) and (9). The photon mass is
very tiny, so if:

$$J_\mu^a = -\epsilon_0 \left(\frac{mc}{\hbar}\right)^2 A_\mu^a \qquad -(86)$$

The vacuum change current density from photon
mass is very tiny, even for a very large
vacuum potential

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