258(6): Some Solutions of the Beltrami Equation

For the free electromagnetic field:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(1)$$

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} . \quad -(2)$$

Therefore:

$$\underline{\nabla} \cdot (\kappa \underline{B}) = \underline{\nabla} \cdot \underline{\nabla} \times \underline{B} = 0 \quad -(3)$$

$$= \underline{B} \cdot \underline{\nabla} \kappa + \kappa \underline{\nabla} \cdot \underline{B}$$

So:

$$\underline{B} \cdot \underline{\nabla} \kappa = 0 \quad -(4)$$

Note that:

$$\kappa \underline{\nabla} \cdot \underline{B} = \kappa \underline{\nabla} \cdot \left(\frac{1}{\kappa} \underline{\nabla} \times \underline{B} \right)$$

$$= \kappa \left(\underline{\nabla} \left(\frac{1}{\kappa} \right) \cdot \underline{\nabla} \times \underline{B} + \frac{1}{\kappa} \underline{\nabla} \cdot \underline{\nabla} \times \underline{B} \right) .$$

$$= \kappa^2 \underline{\nabla} \left(\frac{1}{\kappa} \right) \cdot \underline{B} = 0 \quad -(5)$$

From eqs. (4) and (5):

$$\boxed{\underline{\nabla} \kappa = \kappa^2 \underline{\nabla} \left(\frac{1}{\kappa} \right)} \quad -(6)$$

It is true that $\kappa = \text{constant}$ $-(7)$

is a solution of eq. (6), but Marsh also considers κ to be a function of X, Y and Z.

Consider the case:

$$\frac{\partial \kappa}{\partial Z} = \kappa^2 \frac{\partial}{\partial Z}\left(\frac{1}{\kappa}\right) \quad -(8)$$

then:

$$\kappa = \int \kappa^2 \frac{\partial}{\partial Z}\left(\frac{1}{\kappa}\right) dZ \quad -(9)$$

The formula for integration by parts is:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \quad -(10)$$

so

$$\int \kappa^2 \frac{\partial}{\partial Z}\left(\frac{1}{\kappa}\right) dZ = \kappa - \int \frac{1}{\kappa} \frac{\partial (\kappa^2)}{\partial Z} dZ \quad -(11)$$

From eqs. (9) and (11):

$$\boxed{\int \frac{1}{\kappa} \frac{\partial \kappa^2}{\partial Z} dZ = 0} \quad -(12)$$

A solution of eq. (12) is:

$$\kappa = \text{constant} \quad -(13)$$

However, if it is assumed that the integration is over a well defined interval a to b then:

$$\int_a^b \frac{1}{\kappa} \frac{\partial \kappa^2}{\partial Z} dZ = 0 \quad -(14)$$

A possible solution of eq. (14) is:

3)

$$\boxed{\kappa = \kappa_0 \sin(\kappa_1 Z)} \quad -(15)$$

where κ_0 and κ_1 are constants. From eq. (15):

$$\frac{\partial \kappa^2}{\partial Z} = 2\kappa_0^2 \kappa_1 \sin(\kappa_1 Z)\cos(\kappa_1 Z) \quad -(16)$$

so

$$\int_a^b 2\kappa_0 \kappa_1 \cos(\kappa_1 Z)\, dZ = 0 \quad -(17)$$

i.e.

$$2\kappa_0 \sin(\kappa_1 Z)\Big|_a^b = 0 \quad -(18)$$

or

$$\sin(\kappa_1 b) = \sin(\kappa_1 a) \quad -(19)$$

This is possible when:

$$\boxed{b = a \pm 2\pi n} \quad -(20)$$

<u>Therefore</u> it is <u>possible</u> for <u>κ to depend</u> on <u>X, Y, Z.</u> These solutions are discussed in G. E. Marsh, "Force Free Magnetic Fields" (World Scientific, 1996).

However the case:

$$\kappa = \text{constant} \quad -(21)$$

is soluble and produces many interesting

4)

solutions. These can be graphed and animated by computer. One of these is the Lundquist solution:

$$B_z(r, Z) = (\alpha^2 + \lambda^2)^{1/2} e^{-\lambda Z} J_0 \left((\alpha^2 + \lambda^2)^{1/2} r \right) \quad -(22)$$

$$B_\theta(r, Z) = \alpha e^{-\lambda Z} J_1 \left((\alpha^2 + \lambda^2)^{1/2} r \right) \quad -(23)$$

$$B_r(r, Z) = \lambda e^{-\lambda Z} J_1 \left((\alpha^2 + \lambda^2)^{1/2} r \right) \quad -(24)$$

in cylindrical polar coordinates. Here α and λ are constants and J_0 and J_1 are the zero and first order Bessel functions. These are solutions of:

$$\underline{\nabla} \times \underline{B} = \alpha \underline{B} \quad -(25)$$

Therefore the complete magnetic field is:

$$\underline{B} = B_r \underline{e}_r + B_\theta \underline{e}_\theta + B_Z \underline{k} \quad -(26)$$
$$= B_X \underline{i} + B_Y \underline{j} + B_Z \underline{k}$$

where

$$\underline{e}_r = \underline{i} \cos\theta + \underline{j} \sin\theta \quad -(27)$$

$$\underline{e}_\theta = -\underline{i} \sin\theta + \underline{j} \cos\theta \quad -(28)$$

$$\cos\theta = \frac{X}{r}, \quad \sin\theta = \frac{Y}{r} \quad -(29)$$

$$r = (X^2 + Y^2)^{1/2} \quad -(30)$$

5) Therefore:

$$\underline{B} = \underline{i}\left(B_r \cos\theta - B_\theta \sin\theta\right) + \underline{j}\left(B_r \sin\theta + B_\theta \cos\theta\right) + B_Z \underline{k} \quad -(31)$$

In order for this to be a solution of free space electromagnetism it must be multiplied by $e^{i\omega t}$, so:

$$\underline{B}\,(\text{free field}) = e^{i\omega t}\,\underline{B} \quad -(32)$$

So for free space electromagnetism of wave number κ:

$$B_r = \lambda \cos(\omega t)\, e^{-\lambda Z} J_1\left((\kappa^2+\lambda^2)^{1/2} r\right) \quad -(33)$$

$$B_\theta = \kappa \cos(\omega t)\, e^{-\lambda Z} J_1\left((\kappa^2+\lambda^2)^{1/2} r\right) \quad -(34)$$

$$B_Z = \cos(\omega t)\, e^{-\lambda Z} J_0\left((\kappa^2+\lambda^2)^{1/2} r\right) \quad -(35)$$

In cylindrical polar coordinates the divergence is

$$\underline{\nabla} \cdot \underline{B} = \frac{1}{r} B_r + \frac{\partial B_r}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_Z}{\partial Z} = 0 \quad -(36)$$

So it must be checked by computer algebra that eqns. (22) to (24) agree with eq. (36)

6) These are similar to Reed's solution:

$$\underline{B} = B^{(0)} \left(J_1(\kappa r) \underline{e}_\theta + J_0(\kappa r) \underline{k} \right) \quad -(37)$$

In note 257(6) the θ part was expressed as:

$$\underline{B}_\theta^{(1)} = \frac{B^{(0)}}{4\pi} \int_0^\pi (i\underline{i} + \underline{j}) \exp(i(\omega t + y + \theta)) \, d\tau + \dots \quad -(38)$$

$$\underline{B}_\theta^{(2)} = \frac{B^{(0)}}{4\pi} \int_0^\pi (-i\underline{i} + \underline{j}) \exp(-i(\omega t + y + \theta)) \, d\tau + \dots \quad -(39)$$

where

$$y = \tau - \kappa r \sin \tau \quad -(40)$$

It should also be checked by computer algebra that

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(41)$$

for eq. (37), i.e.

$$\frac{1}{r} \frac{\partial J_1(\kappa r)}{\partial \theta} + \frac{\partial}{\partial Z} J_0(\kappa r) = 0 \quad -(42)$$

Eq. (41) is true because:

$$J_0(\kappa r) = \frac{1}{\pi} \int_0^\pi \cos(\kappa r \sin \tau) \, d\tau \quad -(43)$$

$$J_1(\kappa r) = \frac{1}{\pi} \int_0^\pi \cos(\tau - \kappa r \sin \tau) \, d\tau \quad -(44)$$

7) and do not have any θ or Z dependence.

However, the Lundquist solutions are more complicated and without computer algebra it is not clear ~~that~~ the divergence of $\underline{B}$ vanishes from eqs. (22) to (24).

<u>Suggestion for Animation</u>

It is clear that the component:

$$\underline{B}_\theta = B^{(0)} e^{i\omega t} J_1(\kappa r) \underline{e}_\theta \quad -(45)$$

has the property: $\underline{\nabla} \cdot \underline{B}_\theta = 0 \quad -(46)$

and as shown in note 257(6) it can be expanded as:

$$\underline{B}_\theta^{(1)} = \frac{B^{(0)}}{4\pi} \int_0^\pi \left(i \, \underline{i} \left(e^{i(\omega t + y)} + e^{i(\omega t - y)} \right) \left(e^{i\theta} - e^{-i\theta} \right) + \underline{j} \left(e^{i(\omega t + y)} + e^{i(\omega t - y)} \right) \left(e^{i\theta} + e^{-i\theta} \right) d\tau \right. \quad -(47)$$

where $y = \tau - \kappa r \sin \tau$

so it is suggested to animate eq. (45) and:

$$\underline{B}_Z = B^{(0)} e^{i\omega t} J_0(\kappa r) \underline{k} \quad -(48)$$