258(1): Analysis of Magnetic and Electric Charge / Current Densities in terms of Beltrami Equations.

The magnetic and electric charge / current densities are defined by:

$$\rho_m^{\ a} = \epsilon_0 c \left(\underline{\omega}^a{}_b \cdot \underline{B}^b - \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \right) \quad -(1)$$

$$\underline{J}_m^{\ a} = \epsilon_0 \left(\underline{\omega}^a{}_b \times \underline{E}^b - \omega_0 c \underline{B}^a - c \left(\underline{A}^b \times \underline{R}^a{}_b(\text{orb}) - A_0^b \underline{R}^a{}_b(\text{spin}) \right) \right) \quad -(2)$$

and

$$\rho_E^{\ a} = \epsilon_0 \left(\underline{\omega}^a{}_b \cdot \underline{E}^b - c \underline{A}^b \cdot \underline{R}^a{}_b(\text{orb}) \right) \quad -(3)$$

$$\underline{J}_E^{\ a} = \frac{1}{\mu_0} \left(\underline{\omega}^a{}_b \times \underline{B}^b + \frac{\omega_0}{c} \underline{E}^b - \left(\underline{A}^b \times \underline{R}^a{}_b(\text{spin}) + A_0^b \underline{R}^a{}_b(\text{orb}) \right) \right) \quad -(4)$$

In the absence of magnetic charge / current density:

$$\underline{\omega}^a{}_b \cdot \underline{B}^b = \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad -(5)$$

and in the absence of magnetic charge / current density:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^b = 0 \quad -(6)$$

so

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b = \underline{A}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(7)$$

In this case it is always possible to write:

$$\underline{\nabla} \times \underline{A}^b = \kappa \underline{A}^b \quad -(8)$$

as a Beltrami vector potential.

2) From eqs. (7) and (9):

$$\kappa \underline{\omega}^a{}_b \cdot \underline{A}^b = \underline{A}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad - (9)$$

So

$$\boxed{\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa \underline{\omega}^a{}_b} \quad - (10)$$

If the potential obeys a Beltrami equation the spin connection vector obeys a Beltrami equation.

The magnetic field is defined by:

$$\underline{B}^b = \underline{\nabla} \times \underline{A}^b - \underline{\omega}^b{}_c \times \underline{A}^c \quad - (11)$$

and also obeys a Beltrami equation:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \quad - (12)$$

From eqs. (8) and (11):

$$\underline{B}^b = \kappa \underline{A}^b - \underline{\omega}^b{}_c \times \underline{A}^c \quad - (13)$$

Multiply eq. (13) by $\underline{\omega}^a{}_b$:

$$\kappa \underline{\omega}^a{}_b \cdot \underline{A}^b - \underline{\omega}^a{}_b \cdot \underline{\omega}^b{}_c \times \underline{A}^c = \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad - (14)$$

Now use:

$$\underline{\omega}^a{}_b \cdot \underline{\omega}^b{}_c \times \underline{A}^c = \underline{A}^c \cdot (\underline{\omega}^a{}_b \times \underline{\omega}^b{}_c) \quad - (15)$$

and relabel summation indices to find:

$$\kappa \underline{\omega}^a{}_b \cdot \underline{A}^b - \underline{A}^b \cdot (\underline{\omega}^a{}_c \times \underline{\omega}^c{}_b) = \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad - (16)$$

) It follows that:

$$\underline{R}^a{}_b(\text{spin}) = \kappa\,\underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(17)$$

$$= \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b$$

QED. The analysis correctly and self consistently produces the correct definition of the spin curvature.

In the absence of a magnetic monopole:

$$\underline{\nabla} \cdot \underline{B}^b = 0 = \kappa\,\underline{\nabla} \cdot \underline{A}^b - \underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{A}^c \quad -(18)$$

so

$$\boxed{\underline{\nabla} \cdot \underline{A}^b = 0} \quad -(19)$$

from eqs. (6) and (18). From eqs. (8) and (19):

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^b = 0 \quad -(20)$$

which is a self consistent result.

From eq. (10):

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{\omega}^a{}_b = \kappa\,\underline{\nabla} \cdot \underline{\omega}^a{}_b \quad -(21)$$

$$= 0$$

so

$$\boxed{\underline{\nabla} \cdot \underline{\omega}^a{}_b = 0} \quad -(22)$$

4) From eq. (22) it follows that:

$$\underline{\nabla} \times (\underline{\omega}^a{}_c \times \underline{\omega}^c{}_b) = \underline{\omega}^a{}_c (\underline{\nabla} \cdot \underline{\omega}^c{}_b) - (\underline{\nabla} \cdot \underline{\omega}^a{}_c)\underline{\omega}^c{}_b$$

$$+ (\underline{\omega}^c{}_b \cdot \underline{\nabla})\underline{\omega}^a{}_c - (\underline{\omega}^a{}_c \cdot \underline{\nabla})\underline{\omega}^c{}_b \quad -(23)$$

$$= (\underline{\omega}^c{}_b \cdot \underline{\nabla})\underline{\omega}^a{}_c - (\underline{\omega}^a{}_c \cdot \underline{\nabla})\underline{\omega}^c{}_b$$

From eq. (10):

$$\underline{\omega}^c{}_b \cdot \underline{\nabla} = \frac{1}{\kappa} \underline{\nabla} \times \underline{\omega}^c{}_b \cdot \underline{\nabla}$$

$$= \frac{1}{\kappa} \underline{\omega}^c{}_b \cdot \underline{\nabla} \times \underline{\nabla} \quad -(24)$$

$$= 0$$

Similarly:

$$\underline{\omega}^a{}_c \cdot \underline{\nabla} = 0 \quad -(25)$$

So

$$\boxed{\underline{\nabla} \times (\underline{\omega}^a{}_c \times \underline{\omega}^c{}_b) = \underline{0}} \quad -(26)$$

Therefore:

$$\underline{\nabla} \times \underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times (\underline{\nabla} \times \underline{\omega}^a{}_b)$$

$$= \kappa \, \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(27)$$

so

$$\boxed{\underline{R}^a{}_b(\text{spin}) = \kappa \, \underline{\omega}^a{}_b} \quad -(28)$$

5) and:

$$\boxed{\underline{\nabla} \times \underline{R}^a{}_b(\text{spin}) = \kappa \underline{R}^a{}_b(\text{spin})} \quad -(29)$$

From eqs (17) and (28):

$$\boxed{\underline{\omega}^a{}_c \times \underline{\omega}^c{}_b = \underline{0}} \quad -(30)$$

The magnetic field is defined as:

$$\begin{aligned} \underline{B}^a &= \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(31) \\ &= \kappa \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b . \end{aligned}$$

So:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{\nabla} \times \underline{A}^a - \underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{A}^b \right) \quad -(32)$$

However:

$$\begin{aligned} \underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{A}^b \right) &= \underline{\omega}^a{}_b \left(\underline{\nabla} \cdot \underline{A}^b \right) - \left(\underline{\nabla} \cdot \underline{\omega}^a{}_b \right) \underline{A}^b \\ &\quad + \left(\underline{A}^b \cdot \underline{\nabla} \right) \underline{\omega}^a{}_b - \left(\underline{\omega}^a{}_b \cdot \underline{\nabla} \right) \underline{A}^b \\ &= \left(\underline{A}^b \cdot \underline{\nabla} \right) \underline{\omega}^a{}_b - \left(\underline{\omega}^a{}_b \cdot \underline{\nabla} \right) \underline{A}^b \quad -(33) \end{aligned}$$

because:

$$\underline{\nabla} \cdot \underline{A}^a = 0 \quad -(34)$$

and $$\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa\, \underline{\omega}^a{}_b \quad -(35)$$

so $$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\omega}^a{}_b) = \kappa\, \underline{\nabla} \cdot \underline{\omega}^a{}_b = 0 \quad -(36)$$

so $$\underline{\nabla} \cdot \underline{\omega}^a{}_b = 0 \quad -(37)$$

as in eq. (22). Also:

$$\underline{A}^b \cdot \underline{\nabla} = \frac{1}{\kappa}\, \underline{\nabla} \times \underline{A}^b \cdot \underline{\nabla} = 0 \quad -(38)$$

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} = \frac{1}{\kappa}\, \underline{\nabla} \times \underline{\omega}^a{}_b \cdot \underline{\nabla} = 0 \quad -(39)$$

So:

$$\boxed{\underline{\nabla} \times (\underline{\omega}^a{}_b \times \underline{A}^b) = \underline{0}} \quad -(40)$$

and

$$\boxed{\underline{\nabla} \times \underline{B}^a = \kappa\, \underline{\nabla} \times \underline{A}^a} \quad -(41)$$

It follows that:

$$\boxed{\underline{\nabla} \times \underline{B}^a = \kappa^2 \underline{A}^a = \kappa\, \underline{B}^a} \quad -(42)$$

Q.E.D. Therefore in the absence of a magnetic monopole:

7)

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{A}^a &= \kappa \underline{A}^a \\ \underline{\nabla} \times \underline{B}^a &= \kappa \underline{B}^a \\ \underline{\nabla} \times \underline{\omega}^a{}_b &= \kappa \underline{\omega}^a{}_b \\ \underline{\nabla} \times \underline{R}^a{}_b(\text{spin}) &= \kappa \underline{R}^a{}_b(\text{spin}) \end{aligned}} \quad -(43)$$

These are all eigenvalues of the curl operator and can all generate all the known properties of Beltrami equations.