258(5): Proof of the Free Space Beltrami Equations in ECE Theory.

Start with the Gauss law:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(1)$$

and use the identity:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{B}^a = 0 \quad -(2)$$

It follows that:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \quad -(3)$$

and

$$\underline{B}^a = \frac{1}{\kappa} \underline{\nabla} \times \underline{B}^a \quad -(4)$$

Therefore:

$$\begin{aligned}\underline{\nabla} \cdot \underline{\nabla} \times \underline{B}^a &= \underline{\nabla} \cdot (\kappa \underline{B}^a) \\ &= \kappa \underline{\nabla} \cdot \underline{B}^a + \underline{B}^a \cdot \underline{\nabla} \kappa \\ &= 0\end{aligned} \quad -(5)$$

Therefore:

$$\underline{B}^a \cdot \underline{\nabla} \kappa = 0 \quad -(6)$$

Also:

$$\underline{\nabla} \cdot \underline{B}^a = \underline{\nabla} \cdot \left(\frac{1}{\kappa} \underline{\nabla} \times \underline{B}^a \right) = 0 \quad -(7)$$

so:

$$\begin{aligned}\underline{\nabla} \times \underline{B}^a \cdot \underline{\nabla} \left(\frac{1}{\kappa} \right) &= 0 \\ &= \kappa \underline{B}^a \cdot \underline{\nabla} \left(\frac{1}{\kappa} \right)\end{aligned} \quad -(8)$$

From eqs. (6) and (8):

$$\boxed{\frac{1}{\kappa} \underline{\nabla} \kappa = \underline{\nabla} \left(\frac{1}{\kappa} \right)} \quad -(9)$$

2) If:

$$\kappa = \kappa(X, Y, Z) \quad - (10)$$

it is governed by eq. (9). If κ is a constant then eq. (9) is obeyed automatically. Usually in Beltrami theory, κ is a constant independent of X, Y and Z.

In free space:

$$\underline{\nabla} \cdot \underline{E}^a = 0 \quad - (11)$$

so it follows immediately that:

$$\boxed{\underline{\nabla} \times \underline{E}^a = \kappa \underline{E}^a} \quad - (12)$$

Also in free space:

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{0} \quad - (13)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2} \frac{\partial \underline{E}^a}{\partial t} = \underline{0} \quad - (14)$$

so:

$$\frac{\partial \underline{B}^a}{\partial t} = -\kappa \underline{E}^a \quad - (15)$$

$$\frac{1}{c^2} \frac{\partial \underline{E}^a}{\partial t} = \kappa \underline{B}^a \quad - (16)$$

It follows that:

$$\frac{\partial^2 \underline{E}^a}{\partial t^2} = -\omega^2 \underline{E}^a \quad - (17)$$

3) and
$$\frac{\partial^2 \underline{B}^a}{\partial t^2} = -\omega^2 \underline{B}^a \quad -(18)$$

so
$$\underline{E}^a = \underline{E}_0^a e^{i\omega t} \quad -(19)$$
$$\underline{B}^a = \underline{B}_0^a e^{i\omega t} \quad -(20)$$

in free space, where:
$$\underline{\nabla} \times \underline{E}^a = \kappa \underline{E}^a \quad -(21)$$
$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \quad -(22)$$

Note carefully that eqns (19) - (22) are true for all electromagnetic fields in free space.

From the Cartan identity:
$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^b = 0 \quad -(23)$$

in free space. It follows that:
$$\boxed{\underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{A}^b\right) = \kappa\, \underline{\omega}^a{}_b \times \underline{A}^b} \quad -(24)$$

The solutions of $\underline{\omega}^a{}_b \times \underline{A}^b$ are solutions of a Beltrami equation in the absence of a magnetic monopole.

Eq. (23) implies:

4)
$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b = \underline{A}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(25)$$

The magnetic field is defined by:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(26)$$

From eqns. (3) and (26):

$$\begin{aligned} \underline{\nabla} \times \underline{B}^a &= \underline{\nabla} \times \left(\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \right) \\ &= \kappa \underline{B}^a \\ &= \kappa \left(\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \right) \end{aligned} \quad -(27)$$

So:

$$\underline{\nabla} \times \left(\underline{\nabla} \times \underline{A}^a \right) = \kappa \underline{\nabla} \times \underline{A}^a \quad -(28)$$

i.e

$$\boxed{\underline{\nabla} \times \underline{A}^a = \kappa \underline{A}^a} \quad -(29)$$

From eqns. (25) and (29):

$$\boxed{\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa \underline{\omega}^a{}_b} \quad -(30)$$

Therefore in free space:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \quad -(31)$$

$$\underline{\nabla} \times \underline{E}^a = \kappa \underline{E}^a \quad -(32)$$

$$\underline{\nabla} \times \underline{A}^a = \kappa \underline{A}^a \quad -(33)$$

$$\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa \underline{\omega}^a{}_b \quad -(34)$$

$$\underline{\nabla} \times \left(\underline{\omega}^a{}_b \times \underline{A}^b \right) = \underline{\omega}^a{}_b \times \underline{A}^b \quad -(35)$$

5) These are general results for vacuum electromagnetism of any kind. All these quantities have a rich variety of solutions, all with longitudinal components.

Plane Wave Solutions

In this case:

$$\underline{\omega}^a{}_b = \kappa \epsilon^a{}_{bc} \underline{A}^c / A^{(0)} \quad -(36)$$

where:

$$\underline{A}^{(1)} \times \underline{A}^{(2)} = i A^{(0)} \underline{A}^{(3)*} \quad -(37)$$

$$\underline{A}^{(3)} \times \underline{A}^{(1)} = i A^{(0)} \underline{A}^{(2)*} \quad -(38)$$

$$\underline{A}^{(2)} \times \underline{A}^{(3)} = i A^{(0)} \underline{A}^{(1)*} \quad -(39)$$

Here:

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} \quad -(40)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} \quad -(41)$$

$$\underline{\nabla} \times \underline{A}^{(3)} = 0 \, \underline{A}^{(3)} \quad -(42)$$

and

$$\underline{\nabla} \cdot \underline{A}^{(1)} = \underline{\nabla} \cdot \underline{A}^{(2)} = \underline{\nabla} \cdot \underline{A}^{(3)} = 0 \quad -(43)$$

where

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(44)$$

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} \quad -(45)$$

$$\underline{A}^{(3)} = A^{(0)} \underline{k} \quad -(46)$$

6) The electromagnetic phase is:

$$\phi = \omega t - \kappa Z \quad -(47)$$

where ω is the angular frequency at t and κ the wave vector magnitude at Z.

For plane waves the longitudinal component is a constant along Z.

It is easily checked that eqs. (31) to (35) hold for plane waves. For example:

$$\nabla \times \left(\underline{A}^{(2)} \times \underline{A}^{(3)}\right) = \kappa \underline{A}^{(2)} \times \underline{A}^{(3)} \quad -(48)$$

From eq. (39), this equation is

$$\nabla \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)}. \quad -(49)$$

Self consistently:

$$\begin{aligned}\nabla \times \left(\underline{A}^{(2)} \times \underline{A}^{(3)}\right) &= \underline{A}^{(2)}\left(\nabla \cdot \underline{A}^{(3)}\right) - \left(\nabla \cdot \underline{A}^{(2)}\right)\underline{A}^{(3)} \\ &\quad + \left(\underline{A}^{(2)} \cdot \nabla\right)\underline{A}^{(3)} - \left(\underline{A}^{(3)} \cdot \nabla\right)\underline{A}^{(2)} \quad -(50) \\ &= -\underline{A}^{(0)} \frac{\partial \underline{A}^{(2)}}{\partial Z} \\ &= \kappa \underline{A}^{(2)}\end{aligned}$$

Q.E.D.