255(4): Derivation of the Second Bianchi Identity from the First Bianchi Identity.

The whole of Einstein's general relativity is based on the second Bianchi identity.

Unfortunately this identity is incorrect because of the neglect of torsion. In this note all details are given of the derivation of the second Bianchi identity. In differential form notation the second Bianchi identity is:

$$d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \quad -(1)$$

In tensor notation:

$$\partial_\mu R^a{}_{b\nu\rho} + \partial_\rho R^a{}_{b\mu\nu} + \partial_\nu R^a{}_{b\rho\mu} + \omega^a{}_{\mu c} R^c{}_{b\nu\rho} + \omega^a{}_{\rho c} R^c{}_{\mu\nu} + \omega^a{}_{\mu c} R^c{}_{\nu\rho} = 0 \quad -(2)$$

$$- R^a{}_{c\mu\nu}\,\omega^c{}_{\rho b} - R^a{}_{c\rho\mu}\,\omega^c{}_{\nu b} - R^a{}_{c\nu\rho}\,\omega^c{}_{\mu b} \quad -(3)$$

Note that:

$$D_\mu R^a{}_{b\nu\rho} = \partial_\mu R^a{}_{b\nu\rho} + \omega^a{}_{\mu c} R^c{}_{b\nu\rho} - R^a{}_{c\mu\nu}\,\omega^c{}_{\rho b}$$

and so on, by definition of the covariant derivative of a rank two tensor (S.M. Carroll for example, online notes).

So eq. (2) is:

$$D_\mu R^a{}_{b\nu\rho} + D_\rho R^a{}_{b\mu\nu} + D_\nu R^a{}_{b\rho\mu} = 0 \quad -(4)$$

In eq. (4):

2)

$$R^a{}_{b\mu\nu} = q^a{}_\kappa \, q^\rho{}_b \, R^\kappa{}_{\rho\mu\nu} \quad -(5)$$

and so on. Note that:

$$D_\lambda R^a{}_{b\mu\nu} = D_\lambda R^\kappa{}_{\rho\mu\nu} \quad -(6)$$

because:

$$D_\lambda \left(q^a{}_\kappa \, q^\rho{}_b \right)$$

$$= q^a{}_\kappa D_\lambda q^\rho{}_b + q^\rho{}_b D_\lambda q^a{}_\kappa$$

$$= 0 \quad -(7)$$

because of the tetrad postulate:

$$D_\lambda q^\rho{}_b = D_\lambda q^a{}_\kappa = 0. \quad -(8)$$

So the second Bianchi identity is:

$$D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu}$$

$$= 0 \quad -(9)$$

QED.

The first Bianchi identity is:

$$R^\kappa{}_{\mu\nu\rho} + R^\kappa{}_{\rho\mu\nu} + R^\kappa{}_{\nu\rho\mu} = 0 \quad -(10)$$

which in differential form notation is:

$$q^b \wedge R^a{}_b = 0 \quad -(11)$$

Eq. (9) is a consequence of eq. (10).

3) This is proven as follows:

$$\begin{aligned} & D_\mu \left(R^{\kappa\tau}{}_{\lambda\nu\rho} + R^{\kappa\tau}{}_{\rho\lambda\nu} + R^{\kappa\tau}{}_{\nu\rho\lambda} \right) \\ - & D_\rho \left(R^{\kappa\tau}{}_{\lambda\mu\nu} + R^{\kappa\tau}{}_{\nu\lambda\mu} + R^{\kappa\tau}{}_{\mu\nu\lambda} \right) \\ - & D_\nu \left(R^{\kappa\tau}{}_{\lambda\rho\mu} + R^{\kappa\tau}{}_{\mu\lambda\rho} + R^{\kappa\tau}{}_{\rho\mu\lambda} \right) = 0 \qquad -(12) \end{aligned}$$

Therefore:

$$\begin{aligned} & D_\mu R^{\kappa\tau}{}_{\lambda\nu\rho} + D_\rho R^{\kappa\tau}{}_{\lambda\mu\nu} + D_\nu R^{\kappa\tau}{}_{\lambda\rho\mu} \\ - & \big(D_\mu R^{\kappa\tau}{}_{\rho\lambda\nu} + D_\rho R^{\kappa\tau}{}_{\nu\lambda\mu} + D_\nu R^{\kappa\tau}{}_{\mu\lambda\rho} \\ & + D_\mu R^{\kappa\tau}{}_{\nu\rho\lambda} + D_\rho R^{\kappa\tau}{}_{\mu\nu\lambda} + D_\nu R^{\kappa\tau}{}_{\rho\mu\lambda} \big) \\ & = 0 \qquad -(13) \end{aligned}$$

Using the first Bianchi identity:

$$D_\mu \left(R^{\kappa\tau}{}_{\rho\lambda\nu} + R^{\kappa\tau}{}_{\nu\rho\lambda} \right) = - D_\mu R^{\kappa\tau}{}_{\lambda\nu\rho} \qquad -(14)$$

and so on.

So eq. (13) is:

$$3 \left(D_\mu R^{\kappa\tau}{}_{\lambda\nu\rho} + D_\rho R^{\kappa\tau}{}_{\lambda\mu\nu} + D_\nu R^{\kappa\tau}{}_{\lambda\rho\mu} \right) = 0 \qquad (15)$$

which is the second Bianchi identity, QED

The second Bianchi identity is however incorrect because of the neglect of torsion.

4) The corrected first Bianchi identity is the Cartan identity:

$$D \wedge T^a := q^b \wedge R^a{}_b \quad - (16)$$

Therefore the second Bianchi identity must also be corrected for torsion. This was first done in UFT 88, a heavily studied paper.

From eq. (15) the second Bianchi identity is

$$D_\mu \left(q^b \wedge R^a{}_b \right) = 0 \quad - (17)$$

so the corrected second Bianchi identity is:

$$\boxed{D_\mu \left(D \wedge T^a \right) := D_\mu \left(q^b \wedge R^a{}_b \right)} \quad - (18)$$

In tensor notation:

$$D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} :=$$
$$D_\mu \left(D \wedge T^a \right) \quad - (19)$$

The covariant wedge derivative of T^a is:

$$D \wedge T^a = D_\lambda T^a{}_{\nu\rho} + D_\rho T^a{}_{\lambda\nu} + D_\nu T^a{}_{\rho\lambda}$$
$$- (20)$$

5) Here:

$$T^a{}_{\nu\rho} = q^a{}_\kappa T^\kappa{}_{\nu\rho} \quad -(21)$$

and so on. It follows that:

$$D_\lambda T^a{}_{\nu\rho} = D_\lambda T^\kappa{}_{\nu\rho} \quad -(22)$$

because:

$$D_\lambda q^a{}_\kappa = 0 \quad -(23)$$

by the tetrad postulate.

So eq. (18) is:

$$D_\mu \left(R^\kappa{}_{\lambda\nu\rho} + R^\kappa{}_{\rho\lambda\nu} + R^\kappa{}_{\nu\rho\lambda} \right) := D_\mu \left(D_\lambda T^\kappa{}_{\nu\rho} + D_\rho T^\kappa{}_{\lambda\nu} + D_\nu T^\kappa{}_{\rho\lambda} \right) \quad -(24)$$

Using the same logic as in eqs. (12) to (15) leads to the correct second Bianchi identity:

$$\boxed{\begin{aligned} &D_\mu D_\lambda T^\kappa{}_{\nu\rho} + D_\rho D_\lambda T^\kappa{}_{\mu\nu} + D_\nu D_\lambda T^\kappa{}_{\rho\mu} \\ := \; &D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} \end{aligned}} \quad -(25)$$

Note that eq. (25) is a trivial extension of the Cartan identity:

6)

$$D_\lambda T^{\kappa_1}{}_{\nu\rho} + D_\rho T^{\kappa_1}{}_{\lambda\nu} + D_\nu T^{\kappa_1}{}_{\rho\lambda} := R^{\kappa_1}{}_{\lambda\nu\rho} + R^{\kappa_1}{}_{\rho\lambda\nu} + R^{\kappa_1}{}_{\nu\rho\lambda} \quad - (26)$$

because eq. (25) is derived simply by differentiating both sides of eq. (26) by D_μ. There is no new information in the second Bianchi identity and ECE uses eq. (26).

The old Einstein physics incorrectly assumed that:

$$\Gamma^\lambda_{\mu\nu} =? \Gamma^\lambda_{\nu\mu} \quad - (27)$$

In eq. (27) defined the Christoffel connection. In the old Einstein physics the torsion was zero:

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} =? 0 \quad - (28)$$

In general the connection $\Gamma^\lambda_{\mu\nu}$ is asymmetric in its lower two indices, i.e. has no particular symmetry. Christoffel assumed eq. (27) in an arbitrary way. Neither torsion nor curvature was known to Christoffel in the eighteen sixties. In the mid twentieth century the commutator method showed that:

$$[D_\mu, D_\nu] V^\rho = -\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) D_\lambda V^\rho + R^\rho{}_{\mu\nu\sigma} V^\sigma \quad -(29)$$

The commutator is antisymmetric:

$$[D_\mu, D_\nu] = -[D_\nu, D_\mu], \quad -(30)$$

so

$$\boxed{\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu}} \quad -(31)$$

The torsion is always non-zero. There is a one to one correspondence in eq. (29) between the commutator and the connection:

$$\boxed{[D_\mu, D_\nu] V^\rho = -\Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \ldots} \quad -(32)$$

If the connection were symmetric the commutator would vanish, and the curvature and torsion would both vanish:

$$[D_\mu, D_\nu] V^\rho = 0, \quad -(33)$$

and

$$T^\lambda_{\mu\nu} = 0 \quad -(34)$$

$$R^\rho{}_{\mu\nu\sigma} = 0 \quad -(35)$$

because:

8)

$$\nabla^\sigma \neq 0, \quad -(36)$$

$$D_\lambda \nabla^\rho \neq 0. \quad -(37)$$

On his p. 81 Carroll states clearly that the second Bianchi identity for the general connection must include torsion. Carroll also gives eq. (29), but arbitrarily sets torsion to zero in an illogical way.

The old Einstein physics proceeded by defining the Ricci tensor:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \quad -(38)$$

then assumed a symmetric connection to find from the second Bianchi identity:

$$D^\mu R_{\rho\mu} = \frac{1}{2} D_\rho R \quad -(39)$$

where R is the Ricci scalar. It then defined the Einstein tensor:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad -(40)$$

$$D^\mu G_{\mu\nu} = 0 \quad -(41)$$

Finally it assumed:

$$D^\mu G_{\mu\nu} = k \underbrace{D^\mu N_{\mu\nu}}_{=0} \quad -(42)$$

where $N_{\mu\nu}$ is the canonical energy momentum

9) tensor. The Noether Theorem states that

$$D^\mu N_{\mu\nu} = 0 \quad -(43)$$

which is conservation of energy momentum. Finally Einstein assumed that:

$$\boxed{G_{\mu\nu} = k N_{\mu\nu}} \quad -(44)$$

which is the Einstein field equation.

Unfortunately eq. (44) is incorrect geometrically because it incorrectly assumes a symmetric connection.

In the post Einsteinian era of ECE theory the field equations of gravitation are based on the geometrically correct Cartan identity:

$$D \wedge T^a := q^b \wedge R^a{}_b \quad -(45)$$

and Evans identity:

$$D \wedge \tilde{T}^a := q^b \wedge \tilde{R}^a{}_b. \quad -(46)$$

The second Bianchi identity is not needed and is not used.