Note 255(1): Basic Hodge Dual Definitions.

The basic Hodge dual from magnetic to electric field is:

$$\tilde{F}^{03} = \epsilon^{0312} F_{12}$$
$$\tilde{F}^{01} = \epsilon^{0123} F_{23} \quad -(1)$$
$$\tilde{F}^{02} = \epsilon^{0231} F_{31}$$

where:

$$\epsilon^{0123} = 1 = -\epsilon^{0213} = \epsilon^{0231} = -\epsilon^{0321} = \epsilon^{0312} \quad -(2)$$

So:

$$(\partial_1 A_2 - \partial_2 A_1)_{HD} = \partial^0 A^3 - \partial^3 A^0$$
$$(\partial_2 A_3 - \partial_3 A_2)_{HD} = \partial^0 A^1 - \partial^1 A^0 \quad -(3)$$
$$(\partial_3 A_1 - \partial_1 A_3)_{HD} = \partial^0 A^2 - \partial^2 A^0$$

with the Minkowski metric:

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad -(4)$$

otherwise the metric determinant enters into the Hodge dual.

Here:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right), \quad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad -(5)$$

$$A^\mu = (A^0, \underline{A}) = \left(\frac{\phi}{c}, \underline{A}\right) \quad -(6)$$

$$A_\mu = (A^0, -\underline{A}) = \left(\frac{\phi}{c}, -\underline{A}\right) \quad -(7)$$

So

$$\partial_1 A_2 - \partial_2 A_1 = -\left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right) = -B_Z$$
$$\partial_3 A_1 - \partial_1 A_3 = -\left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right) = -B_Y \quad -(8)$$
$$\partial_2 A_3 - \partial_3 A_2 = -\left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right) = -B_{\cancel{Z}X}$$

2)

$$\partial^0 A^3 - \partial^3 A^0 = \frac{1}{c}\frac{\partial A_Z}{\partial t} + \frac{\partial A^0}{\partial Z} = -\frac{E_Z}{c}$$

$$\partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_X}{\partial t} + \frac{\partial A^0}{\partial X} = -\frac{E_X}{c} \quad -(9)$$

$$\partial^0 A^2 - \partial^2 A^0 = \frac{1}{c}\frac{\partial A_Y}{\partial t} + \frac{\partial A^0}{\partial Y} = -\frac{E_Y}{c}$$

Here:

$$\underline{B} = \underline{\nabla} \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ A_X & A_Y & A_Z \end{vmatrix} \quad -(10)$$

$$= \underline{i}\left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right) - \underline{j}\left(\frac{\partial A_Z}{\partial X} - \frac{\partial A_X}{\partial Z}\right) + \underline{k}\left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right)$$

So:

$$\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} = c\left(\underline{\nabla} \times \underline{A}\right)_{HD}} \quad -(11)$$

The Hodge dual transform is:

$$\boxed{\underline{B} \rightarrow \underline{E}/c} \quad -(12)$$

i.e.

$$\underline{E} = c\,\underline{B}_{HD} \quad -(13)$$

The duality transform of the free field is:

$$\underline{E} = ic\underline{B} \quad -(14)$$

3) so

$$\underline{B}_{HD} = i\underline{B} \quad -(15)$$

in free space. Also in free space:

$$\underline{E} = ic\,\underline{\nabla} \times \underline{A} = c\,\underline{\nabla} \times (i\underline{A}) = c\,(\underline{\nabla} \times \underline{A})_{HD} \quad -(16)$$

So

$$(\underline{\nabla} \times \underline{A})_{HD} = \underline{\nabla} \times (i\underline{A}) \quad -(17)$$

and

$$\underline{A}_{HD} = i\underline{A} \quad -(18)$$

Therefore:

$$\boxed{A^{\mu}_{HD} = iA^{\mu}} \quad -(19)$$

in free space.

The basic Hodge dual from electric to magnetic field is:

$$\begin{aligned} \partial^1 A^2 - \partial^2 A^1 &= \epsilon^{1203}(\partial_0 A_3 - \partial_3 A_0) \\ \partial^3 A^1 - \partial^1 A^3 &= \epsilon^{3102}(\partial_0 A_2 - \partial_2 A_0) \\ \partial^2 A^3 - \partial^3 A^2 &= \epsilon^{2301}(\partial_0 A_1 - \partial_1 A_0) \end{aligned} \quad -(20)$$

where:

$$\epsilon^{0123} = -\epsilon^{1023} = \epsilon^{1203} = \epsilon^{0312} = -\epsilon^{3012} = \epsilon^{3102} = \epsilon^{0231} = -\epsilon^{2031} = \epsilon^{2301} = 1 \quad -(21)$$

Therefore:

4) $(\partial_0 A_3 - \partial_3 A_0)_{HD} = \partial^1 A^2 - \partial^2 A^1$

$(\partial_0 A_2 - \partial_2 A_0)_{HD} = \partial^3 A^1 - \partial^1 A^3 \quad -(22)$

$(\partial_0 A_1 - \partial_1 A_0)_{HD} = \partial^2 A^3 - \partial^3 A^2$

i.e.

$$\underline{E}_{HD} = -c\underline{B} \quad -(23)$$

and

$$\underline{B} = -\underline{E}_{HD}/c \quad -(24)$$

The complete duality transform is:

$$\boxed{\begin{aligned}\underline{B}_{HD} &= \underline{E}/c \\ \underline{E}_{HD} &= -c\underline{B}\end{aligned}} \quad -(25)$$

For free fields:

$$\boxed{\begin{aligned}\underline{E} &= ic\underline{B} = c\underline{B}_{HD} \quad &-(26)\\ \underline{B} &= -\frac{i\underline{E}}{c} = -\frac{\underline{E}_{HD}}{c} \quad &-(27)\end{aligned}}$$

therefore for free fields:

$$\underline{B}_{HD} = i\underline{B} \quad -(28)$$

$$\underline{E}_{HD} = i\underline{E} \quad -(29)$$

In general:

5)
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad -(30)$$

where:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_X & -cB_Y & -cB_Z \\ cB_X & 0 & E_Z & -E_Y \\ cB_Y & -E_Z & 0 & E_X \\ cB_Z & E_Y & -E_X & 0 \end{bmatrix}, \quad F_{\rho\sigma} = \begin{bmatrix} 0 & E_X & E_Y & E_Z \\ -E_X & 0 & -cB_Z & cB_Y \\ -E_Y & cB_Z & 0 & -cB_X \\ -E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \quad -(31)$$

Type Two Hodge Duality

This is defined by the relation between the two tensors of eq. (31):

$$\begin{aligned}
\tilde{F}^{01} &= \epsilon^{0123}F_{23}, & -cB_X &= -cB_X, & \epsilon^{0123} &= 1 \\
\tilde{F}^{02} &= \epsilon^{0231}F_{31}, & -cB_Y &= -cB_Y, & \epsilon^{0231} &= 1 \\
\tilde{F}^{03} &= \epsilon^{0312}F_{12}, & -cB_Z &= -cB_Z, & \epsilon^{0312} &= 1 \\
\tilde{F}^{12} &= \epsilon^{1230}F_{30}, & E_Z &= E_Z, & \epsilon^{1230} &= -1 \\
\tilde{F}^{13} &= \epsilon^{1302}F_{02}, & -E_Y &= -E_Y, & \epsilon^{1302} &= -1 \\
\tilde{F}^{23} &= \epsilon^{2301}F_{01}, & E_X &= E_X, & \epsilon^{2301} &= 1
\end{aligned} \quad -(32)$$

It is seen that the type two Hodge duality has the effect of rearranging elements. It takes electric field components to give the same components in different places. Similarly for the magnetic components. The type one Hodge duality already discussed transforms between electric and magnetic fields

6) Homogeneous Field Equation

This is
$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad -(33)$$

For $\nu = 0$
$$\partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30} = 0 \quad -(34)$$

i.e.
$$\frac{\partial B_X}{\partial X} + \frac{\partial B_Y}{\partial Y} + \frac{\partial B_Z}{\partial Z} = 0 \quad -(35)$$

or
$$\boxed{\underline{\nabla} \cdot \underline{B} = 0} \quad -(36)$$

at the $u(1)$ level.

This is the Gauss law of magnetism

For $\nu = 1$
$$\partial_0 \tilde{F}^{01} + \partial_2 \tilde{F}^{21} + \partial_3 \tilde{F}^{31} = 0 \quad -(37)$$

i.e.
$$-\frac{\partial B_X}{\partial t} - \frac{\partial E_Z}{\partial Y} + \frac{\partial E_Y}{\partial Z} = 0 \quad -(38)$$

$-(39)$

Note that:
$$\underline{\nabla} \times \underline{E} = \underline{i}\left(\frac{\partial E_Z}{\partial Y} - \frac{\partial E_Y}{\partial Z}\right) - \underline{j}\left(\frac{\partial E_Z}{\partial X} - \frac{\partial E_X}{\partial Z}\right) + \underline{k}\left(\frac{\partial E_Y}{\partial X} - \frac{\partial E_X}{\partial Y}\right)$$

So
$$\left(\underline{\nabla} \times \underline{E}\right)_X = \frac{\partial E_Z}{\partial Y} - \frac{\partial E_Y}{\partial Z} \quad -(40)$$

Eq. (38) is therefore:
$$-\frac{\partial B_X}{\partial t} - \left(\underline{\nabla} \times \underline{E}\right)_X = 0 \quad -(41)$$

For $\nu = 2$
$$\partial_0 \tilde{F}^{02} + \partial_1 \tilde{F}^{12} + \partial_3 \tilde{F}^{32} = 0 \quad -(42)$$

i.e

$$-\frac{\partial B_Y}{\partial t} + \frac{\partial E_Z}{\partial X} - \frac{\partial E_X}{\partial Z} = 0 \quad -(43)$$

or

$$-\frac{\partial B_Y}{\partial t} - (\underline{\nabla} \times \underline{E})_Y = 0 \quad -(44)$$

For $\nu = 3$

$$\partial_0 \tilde{F}^{03} + \partial_1 \tilde{F}^{13} + \partial_2 \tilde{F}^{23} = 0 \quad -(45)$$

i.e

$$-\frac{\partial B_Z}{\partial t} - \frac{\partial E_Y}{\partial X} + \frac{\partial E_X}{\partial Y} = 0 \quad -(46)$$

or

$$-\frac{\partial B_Z}{\partial t} - (\underline{\nabla} \times \underline{E})_Z = 0 \quad -(47)$$

Eqs. (40), (44) and (47) can be summarized as:

$$-\frac{\partial \underline{B}}{\partial t} - \underline{\nabla} \times \underline{E} = \underline{0} \quad -(48)$$

i.e

$$\boxed{\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = \underline{0}} \quad -(49)$$

This is the Faraday law of induction at the u(1) level. Note carefully that the actual result is eq. (48), with negative signs.

Free Space Inhomogeneous Field Equation

This is $\partial_\mu F^{\mu\nu} = 0 \quad -(50)$

The field tensor with raised indices is:

8)

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} \quad -(51)$$

where $g^{\mu\rho} = g^{\nu\sigma} = \mathrm{diag}(1, -1, -1, -1) \quad -(52)$

so

$$F^{01} = g^{00} g^{11} F_{01} = -F_{01} \quad -(53)$$

$$F^{02} = g^{00} g^{22} F_{02} = -F_{02} \quad -(54)$$

$$F^{03} = g^{00} g^{33} F_{03} = -F_{03} \quad -(55)$$

$$F^{12} = g^{11} g^{22} F_{12} = F_{12} \quad -(56)$$

$$F^{13} = g^{11} g^{33} F_{13} = F_{13} \quad -(57)$$

$$F^{23} = g^{22} g^{33} F_{23} = F_{23} \quad -(58)$$

From eqs. (31) and (51):

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \quad -(59)$$

<u>For $\nu = 0$</u>

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = 0 \quad -(60)$$

i.e

$$\frac{\partial E_X}{\partial X} + \frac{\partial E_Y}{\partial Y} + \frac{\partial E_Z}{\partial Z} = 0 \quad -(61)$$

or

$$\boxed{\underline{\nabla} \cdot \underline{E} = 0} \quad -(62)$$

<u>This is the free space Coulomb law at the U(1) level.</u>

9) For $\nu = 1$

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = 0 \quad -(63)$$

i.e.

$$-\frac{1}{c}\frac{\partial E_X}{\partial t} + c\frac{\partial B_Z}{\partial Y} - c\frac{\partial B_Y}{\partial Z} = 0 \quad -(64)$$

or

$$(\underline{\nabla} \times \underline{B})_X - \frac{1}{c^2}\frac{\partial E_X}{\partial t} = 0 \quad -(65)$$

For $\nu = 2$

$$\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} = 0 \quad -(66)$$

i.e.

$$-\frac{1}{c}\frac{\partial E_Y}{\partial t} - c\frac{\partial B_Z}{\partial X} + c\frac{\partial B_X}{\partial Z} = 0 \quad -(67)$$

or

$$-\frac{1}{c}\frac{\partial E_Y}{\partial t} + (\underline{\nabla} \times \underline{B})_Y = 0 \quad -(68)$$

For $\nu = 3$

$$\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = 0 \quad -(69)$$

i.e.

$$-\frac{1}{c}\frac{\partial E_Z}{\partial t} + c\frac{\partial B_Y}{\partial X} - c\frac{\partial B_X}{\partial Y} = 0 \quad -(70)$$

or

$$-\frac{1}{c}\frac{\partial E_Z}{\partial t} + (\underline{\nabla} \times \underline{B})_Z = 0 \quad -(71)$$

Eqs. (65), (68) and (71) can be summarized as:

$$\boxed{\underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \underline{0}} \quad -(72)$$

which is the free space Ampère Maxwell law on the U(1) level.

Therefore in summary:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \rightarrow \quad \underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \qquad -(73)$$

$$\partial_\mu F^{\mu\nu} = 0 \rightarrow \quad \underline{\nabla} \cdot \underline{E} = 0, \quad \underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \underline{0} \qquad -(74)$$

Under the type one Hodge dual transformations (28) and (29) these equations are invariant. They are also invariant under the duality transforms:

$$\underline{E} \rightarrow ic\underline{B} \ ; \ \underline{B} \rightarrow -i\frac{\underline{E}}{c} \qquad -(75)$$

Under the type one Hodge dual transforms (53) to (58) they are transformed into each other:

$$\partial_\mu \tilde{F}^{\mu\nu} \rightleftarrows \partial_\mu F^{\mu\nu} \qquad -(76)$$

These symmetries are broken in the presence of matter, when the inhomogeneous equation becomes:

$$\partial_\mu F^{\mu\nu} = j^\nu / \epsilon_0 \qquad -(77)$$