255(8): Vector Definition of Charge Density and Current Density in ECE Theory.

The charge current density is defined by:

$$j^{a\nu} = q^b_{\ \mu} R^a_{\ b}{}^{\mu\nu} - \omega^a_{\ \mu b} T^{b\mu\nu} \quad -(1)$$

The charge density is defined by:

$$j^{a0} = q^b_{\ \mu} R^a_{\ b}{}^{\mu 0} - \omega^a_{\ \mu b} T^{b\mu 0} \quad -(2)$$

Define:

$$\underline{q}^b = q^b_{\ X}\,\underline{i} + q^b_{\ Y}\,\underline{j} + q^b_{\ Z}\,\underline{k} \quad -(3)$$

and note that:

$$q^b_{\ 1} = -q^b_{\ X} \quad \text{etc.} \quad -(4)$$

It follows that:

$$\boxed{j^{a0} = \underline{\omega}^a_{\ b} \cdot \underset{\sim}{T}^b(\text{orb}) - \underline{q}^b \cdot \underline{R}^a_{\ b}(\text{orb})} \quad -(5)$$

where:

$$\underline{\nabla} \cdot \underset{\sim}{T}^a(\text{orb}) = j^{a0} \quad -(6)$$

In these equations:

$$\underset{\sim}{T}^a(\text{orb}) = T^a_{\ X}(\text{orb})\,\underline{i} + T^a_{\ Y}(\text{orb})\,\underline{j} + T^a_{\ Z}(\text{orb})\,\underline{k} \quad -(7)$$

$$\underline{R}^a_{\ b}(\text{orb}) = R^a_{\ bX}(\text{orb})\,\underline{i} + R^a_{\ bY}(\text{orb})\,\underline{j} + R^a_{\ bZ}(\text{orb})\,\underline{k} \quad -(8)$$

2)
$$T^a_x(\text{orb}) = T^{a1}(\text{orb}) = T^{a10} = -T^{a01} \quad -(9)$$
$$T^a_y(\text{orb}) = T^{a2}(\text{orb}) = T^{a20} = -T^{a02} \quad -(10)$$
$$T^a_z(\text{orb}) = T^{a3}(\text{orb}) = T^{a30} = -T^{a03} \quad -(11)$$
$$R^a{}_{bx}(\text{orb}) = R^a{}_b{}^1(\text{orb}) = R^a{}_b{}^{10} = -R^a{}_b{}^{01} \quad -(12)$$
$$R^a{}_{by}(\text{orb}) = R^a{}_b{}^2(\text{orb}) = R^a{}_b{}^{20} = -R^a{}_b{}^{02} \quad -(13)$$
$$R^a{}_{bz}(\text{orb}) = R^a{}_b{}^3(\text{orb}) = R^a{}_b{}^{30} = -R^a{}_b{}^{03} \quad -(14)$$

and
$$\underline{\omega}^a{}_b = \omega^a{}_{xb}\,\underline{i} + \omega^a{}_{yb}\,\underline{j} + \omega^a{}_{zb}\,\underline{k} \quad -(15)$$

To translate into electromagnetic charge density ($C m^{-3}$) use:
$$\underline{E}^a = cA^{(0)}\,\underset{\sim}{T}^a \quad -(16)$$
where $\underline{E}^a$ is the electric field strength in S.I. units of
$$E^a = V m^{-1} = J C^{-1} m^{-1} \quad -(17)$$
and
$$A^{(0)} = J s C^{-1} m^{-1} \quad -(18)$$

It follows that:
$$\boxed{\underline{\nabla} \cdot \underline{E}^a = cA^{(0)} j^{a0} = \frac{\rho^a}{\epsilon_0}} \quad -(19)$$

where ρ^a is the electromagnetic charge density for each index a of $\underline{E}^a$.

3) The units of ϵ_0, the vacuum permittivity, are:

$$\epsilon_0 = J^{-1} C^2 m^{-1} \quad -(20)$$

Therefore:

$$\rho^a = \epsilon_0 c A^{(0)} j^{a0} = \frac{A^{(0)}}{\mu_0 c} j^{a0} \quad -(21)$$

where:

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \quad -(22)$$

The units of μ_0, the vacuum permeability, are:

$$\mu_0 = J s^2 C^{-2} m^{-1} \quad -(23)$$

Therefore:

$$\boxed{\rho^a = \epsilon_0 c A^{(0)} \left(\underline{\omega}^a{}_b \cdot \underset{\sim}{T}^b(\mathrm{orb}) - \underline{q}^b \cdot \underline{R}^a{}_b(\mathrm{orb}) \right)} \quad -(24)$$

and

$$\underline{\nabla} \cdot \underline{E}^a = \rho^a / \epsilon_0 \quad -(25)$$

Now use:

$$\underline{E}^a = c A^{(0)} \underset{\sim}{T}^a(\mathrm{orb}) \quad -(26)$$

$$\underline{A}^b = A^{(0)} \underline{q}^b \quad -(27)$$

to find:

$$\boxed{\rho^a = \epsilon_0 \left(\underline{\omega}^a{}_b \cdot \underline{E}^b - c \underline{A}^b \cdot \underline{R}^a{}_b(\mathrm{orb}) \right)} \quad -(28)$$

4) It follows that:

$$\underline{\omega}^a{}_b \cdot \underline{E}^b = \frac{\rho^a}{\epsilon_0} + c\underline{A}^b \cdot \underline{R}^a{}_b(\text{orb}) \quad -(29)$$

$$= \underline{\nabla} \cdot \underline{E}^a + c\underline{A}^b \cdot \underline{R}^a{}_b(\text{orb})$$

Therefore:

$$\boxed{\begin{aligned}\underline{\nabla} \cdot \underline{E}^a &= \underline{\omega}^a{}_b \cdot \underline{E}^b - c\underline{A}^b \cdot \underline{R}^a{}_b(\text{orb}) \\ &= cA^{(0)}\left(\underline{\omega}^a{}_b \cdot \underset{\sim}{\underline{T}}^b(\text{orb}) - \underline{q}^b \cdot \underline{R}^a{}_b(\text{orb})\right) \quad -(30)\end{aligned}}$$

This is the fundamental equation of electrostatics.

For the free field:

$$\underline{\nabla} \cdot \underline{E}^a = 0 \quad -(31)$$

so

$$\underline{\omega}^a{}_b \cdot \underline{E}^b = c\underline{A}^b \cdot \underline{R}^a{}_b(\text{orb}) \quad -(32)$$

where:

$$\underline{E}^a = -c\underline{\nabla}A^a_0 - \frac{\partial \underline{A}^a}{\partial t} - c\omega^a{}_{0b}\underline{A}^b + cA^b_0\underline{\omega}^a{}_b \quad -(33)$$

Eq. (24) shows that there is an internal geometrical structure to charge density.

5) Experimentally, the electric field strength between two charges is observed to be:

$$\underline{E} = - \frac{e}{4\pi\epsilon_0 r^3} \underline{r} \quad - (34)$$

So

$$E_r = - \frac{e}{4\pi\epsilon_0 r^2} \quad - (35)$$

If it is assumed that:

$$\underline{A}^b = \underline{0} \quad - (36)$$

when considering the electric field between two charges then eq. (30) is:

$$\underline{\nabla} \cdot \underline{E}^a = \underline{\omega}^a{}_b \cdot \underline{E}^b \quad - (37)$$

There is only one sense of radially directed polarization so:

$$\frac{\partial E_r}{\partial r} = \omega E_r \quad - (38)$$

$$= \frac{e}{2\pi\epsilon_0 r^3} = - \frac{\omega e}{4\pi\epsilon_0 r^2}$$

i.e

$$\boxed{\omega = - \frac{2}{r}} \quad - (39)$$

This is one simple example of the use of eq. (30).

In order to find the vectorial structure of current density other terms of eq. (1) need to be considered.

For $\nu = 1$:

$$j^{a1} = q^b{}_\mu R^a{}_b{}^{\mu 1} - \omega^a{}_{\mu b} T^{b \mu 1}$$

$$= q^b{}_0 R^a{}_b{}^{01} + q^b{}_2 R^a{}_b{}^{21} + q^b{}_3 R^a{}_b{}^{31}$$

$$- \omega^a{}_{0b} T^{b01} - \omega^a{}_{2b} T^{b21} - \omega^a{}_{3b} T^{b31} \quad -(40)$$

Here:

$$\underline{T}^a(\text{spin}) = T^a_x(\text{spin})\,\underline{i} + T^a_y(\text{spin})\,\underline{j} + T^a_z(\text{spin})\,\underline{k} \quad -(41)$$

$$\underline{R}^a{}_b(\text{spin}) = R^a{}_{xb}(\text{spin})\,\underline{i} + R^a{}_{yb}(\text{spin})\,\underline{j} + R^a{}_{yb}(\text{spin})\,\underline{k} \quad -(42)$$

Here

$$T^a_x(\text{spin}) = T^{a1}(\text{spin}) = -T^{a23} = T^{a32}$$

$$T^a_y(\text{spin}) = T^{a2}(\text{spin}) = -T^{a31} = T^{a13}$$

$$T^a_z(\text{spin}) = T^{a3}(\text{spin}) = -T^{a12} = T^{a21}$$

$$R^a{}_{xb}(\text{spin}) = R^a{}_b{}^1(\text{spin}) = -R^a{}_b{}^{23} = R^a{}_b{}^{32}$$

$$R^a{}_{yb}(\text{spin}) = R^a{}_b{}^2(\text{spin}) = -R^a{}_b{}^{31} = R^a{}_b{}^{13}$$

$$R^a{}_{zb}(\text{spin}) = R^a{}_b{}^3(\text{spin}) = -R^a{}_b{}^{12} = R^a{}_b{}^{21} \quad -(43)$$

So eq. (40) is:

$$j^{a1} = -q^b{}_0 R^a{}_b{}^1 + q^b{}_2 R^a{}_b{}^3 - q^b{}_3 R^a{}_b{}^2$$

$$+ \omega^a{}_{0b} T^{b1} - \omega^a{}_{2b} T^{b3} + \omega^a{}_{3b} T^{b2} \quad -(44)$$

7) i.e.

$$j^a_X = -q v^b_0 R^a{}_{bX}(\text{orb}) - q v^b_Y R^a{}_{bZ}(\text{spin}) + q v^b_Z R^a{}_{bY}(\text{spin})$$
$$+ \omega^a{}_{0b} T^b{}_X(\text{orb}) + \omega^a{}_{Yb} T^b{}_Z(\text{spin}) - \omega^a{}_{Zb} T^b{}_Y(\text{spin}) \quad -(45)$$

So:

$$\boxed{\underline{j}^a = \omega^a{}_{0b}\underline{T}^b(\text{orb}) + \underline{\omega}^a{}_b \times \underline{T}^b(\text{spin}) - \left(q v^b_0 \underline{R}^a{}_b(\text{orb}) + q \underline{v}^b \times \underline{R}^a{}_b(\text{spin})\right)} \quad -(46)$$

with

$$\underline{\nabla} \times \underline{T}^a(\text{spin}) - \frac{1}{c}\frac{\partial \underline{T}^a(\text{orb})}{\partial t} = \underline{j}^a \quad -(47)$$

Multiplying eq. (47) by $A^{(0)}$ gives:

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \quad -(48)$$
$$= A^{(0)} \underline{j}^a$$

so the electromagnetic charge current density is:

$$\underline{J}^a = \frac{A^{(0)}}{\mu_0}\underline{j}^a \quad (s^{-1} m^{-2} \quad -(49)$$

The electromagnetic four-current is:

$$J^{\mu a} = (c\rho^a, \underline{J}^a) \quad -(50)$$

) where:

$$\rho^a = \epsilon_0 c A^{(0)} \left(\underline{\omega}^a{}_b \cdot \underset{\sim}{T}^b(\text{orb}) - \underline{q}^b \cdot \underline{R}^a{}_b(\text{orb}) \right) \quad -(51)$$

and

$$\underline{J}^a = \epsilon_0 c^2 A^{(0)} \Big(\omega^a{}_b \underset{\sim}{T}^b(\text{orb}) - q^b \cdot \underline{R}^a{}_b(\text{orb})$$
$$+ \underline{\omega}^a{}_b \times \underset{\sim}{T}^b(\text{spin}) - \underline{q}^b \times \underline{R}^a{}_b(\text{spin}) \Big) \quad -(52)$$

Here:

$$\underset{\sim}{T}^b(\text{spin}) = \underline{\nabla} \times \underline{q}^b - \underline{\omega}^a{}_b \times \underline{q}^b \quad -(53)$$

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(54)$$

Therefore:

$$\underline{\omega}^a{}_b \times \underset{\sim}{T}^b(\text{spin}) - \underline{q}^b \times \underline{R}^a{}_b(\text{spin})$$
$$= \underline{\omega}^a{}_b \times (\underline{\nabla} \times \underline{q}^b) - \underline{q}^b \times (\underline{\nabla} \times \underline{\omega}^a{}_b) \quad -(55)$$

Electromagnetic charge - current density is a geometrical property and exists in the vacuum.