255(6): Vector Equations of the Cartan Identity

The Cartan identity is the antisymmetrized tensorial identity:

$$D_\mu T^a_{\nu\rho} + D_\rho T^a_{\mu\nu} + D_\nu T^a_{\rho\mu} := R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} \quad -(1)$$

which, in its Hodge dual representation is:

$$D_\mu \tilde{T}^{\mu\nu} := \tilde{R}_\mu{}^{a\mu\nu} \quad -(2)$$

i.e.

$$\partial_\mu \tilde{T}^{a\mu\nu} + \omega^a_{\mu b} \tilde{T}^{b\mu\nu} := \tilde{R}_\mu{}^{a\mu\nu} \quad -(3)$$

In ECE theory the geometrical structure of the homogeneous field equation is given by:

$$\partial_\mu \tilde{T}^{a\mu\nu} = 0, \quad -(4)$$

so

$$\tilde{R}_\mu{}^{a\mu\nu} = \omega^a_{\mu b} \tilde{T}^{b\mu\nu} \quad -(5)$$

For the purposes of engineering and computation these equations are translated into vector format as follows.

Eq. (4) is:

$$\partial_0 \tilde{T}^{a0\nu} + \partial_1 \tilde{T}^{a1\nu} + \partial_2 \tilde{T}^{a2\nu} + \partial_3 \tilde{T}^{a3\nu} = 0 \quad -(6)$$

i.e. for $\nu = 0, 1, 2, 3$:

$$\partial_1 \tilde{T}^{a10} + \partial_2 \tilde{T}^{a20} + \partial_3 \tilde{T}^{a30} = 0 \quad -(7)$$

$$\partial_0 \tilde{T}^{a01} + \partial_2 \tilde{T}^{a21} + \partial_3 \tilde{T}^{a31} = 0 \quad -(8)$$

$$\partial_0 \tilde{T}^{a02} + \partial_1 \tilde{T}^{a12} + \partial_3 \tilde{T}^{a32} = 0 \quad -(9)$$

$$\partial_0 \tilde{T}^{a03} + \partial_1 \tilde{T}^{a13} + \partial_2 \tilde{T}^{a23} = 0 \quad -(10)$$

2) The Hodge dual of the antisymmetric torsion tensor must now be defined. In note 255(1) eq. (31) the electromagnetic field tensor and its Hodge dual were defined as follows for each a:

$$F_{\rho\sigma} = \begin{bmatrix} 0 & E_X & E_Y & E_Z \\ -E_X & 0 & -cB_Z & cB_Y \\ -E_Y & cB_Z & 0 & -cB_X \\ -E_Z & -cB_Y & cB_X & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{bmatrix} \quad -(11)$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_X & -cB_Y & -cB_Z \\ cB_X & 0 & E_Z & -E_Y \\ cB_Y & -E_Z & 0 & E_X \\ cB_Z & E_Y & -E_X & 0 \end{bmatrix} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix} \quad -(12)$$

where
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad -(13)$$

A convention must be adopted for the definition of the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$. In field theory the convention is usually:

$$\epsilon^{0123} = 1 \quad -(14)$$

As explained by Carroll, the Levi Civita symbol is not a tensor. To convert it to a tensor the following definition is used:

$$\bar{\epsilon}_{\mu\nu\rho\sigma} = |g|^{1/2}\epsilon_{\mu\nu\rho\sigma} \quad -(15)$$

where
$$|g| = \| g_{\mu\nu} \| \quad -(16)$$

) where $|g_{\mu\nu}|$ is the determinant of the metric. Therefore $|g|$ is the modulus or real positive value of the determinant of the metric. For the Minkowski metric:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(17)$$

so

$$|g| = 1 \quad -(18)$$

However, ECE theory is developed for the general metric, so in general:

$$|g| \neq 1. \quad -(19)$$

Similarly:

$$\bar{\epsilon}^{\mu\nu\rho\sigma} = \frac{\text{sgn}(g)}{|g|^{1/2}} \epsilon^{\mu\nu\rho\sigma} \quad -(20)$$

where the signum function is defined by:

$$\begin{aligned} \text{sgn}(g) &= -1 \text{ if } g < 0 \\ &= 0 \text{ if } g = 0 \\ &= 1 \text{ if } g > 0 \end{aligned} \quad -(21)$$

For the Minkowski metric:

$$\frac{\text{sgn}(g)}{|g|^{1/2}} = 1 \quad -(22)$$

For the general metric this is no longer true.

However as shown in previous work the complications introduced by eqs. (15) or (20) do

4) not affect the Cartan identity and its Hodge dual identity because the same factor $|g|^{1/2}$ or $\mathrm{sgn}(g)/|g|^{1/2}$ is used on both sides in deriving eq. (2) from eq. (1). Also by metric compatibility:

$$D_\rho g_{\mu\nu} = 0 \quad -(23)$$

so

$$D_\rho |g|^{1/2} = 0 \quad -(23)$$

and

$$D_\rho \left( \frac{\mathrm{sgn}(g)}{|g|^{1/2}} \right) = 0 \quad -(24).$$

Also in previous work the magnetic part of the electromagnetic field tensor arises from the spin torsion, and the electric part from the orbital torsion. Denote the orbital torsion by $T(\mathrm{orb})$ and the spin torsion by $T(\mathrm{spin})$. Therefore for each $a$:

$$T_{\rho\sigma} = \begin{bmatrix} 0 & -T_1(\mathrm{orb}) & -T_2(\mathrm{orb}) & -T_3(\mathrm{orb}) \\ T_1(\mathrm{orb}) & 0 & T_3(\mathrm{spin}) & -T_2(\mathrm{spin}) \\ T_2(\mathrm{orb}) & -T_3(\mathrm{spin}) & 0 & T_1(\mathrm{spin}) \\ T_3(\mathrm{orb}) & T_2(\mathrm{spin}) & -T_1(\mathrm{spin}) & 0 \end{bmatrix} \quad -(25)$$

$$\tilde{T}^{\mu\nu} = \begin{bmatrix} 0 & -T^1(\mathrm{spin}) & -T^2(\mathrm{spin}) & -T^3(\mathrm{spin}) \\ T^1(\mathrm{spin}) & 0 & T^3(\mathrm{orb}) & -T^2(\mathrm{orb}) \\ T^2(\mathrm{spin}) & -T^3(\mathrm{orb}) & 0 & T^1(\mathrm{orb}) \\ T^3(\mathrm{spin}) & T^2(\mathrm{orb}) & -T^1(\mathrm{orb}) & 0 \end{bmatrix}$$

5) The spin tetrad vector is for each a:

$$\underline{T}(\text{spin}) = T^1(\text{spin})\,\underline{i} + T^2(\text{spin})\,\underline{j} + T^3(\text{spin})\,\underline{k} \quad -(27)$$

and the orbital tetrad vector is:

$$\underline{T}(\text{orb}) = T^1(\text{orb})\,\underline{i} + T^2(\text{orb})\,\underline{j} + T^3(\text{orb})\,\underline{k} \quad -(28)$$

By definition, from eq. (26):

$$T^1(\text{spin}) = \tilde{T}^{10} = -\tilde{T}^{01}$$
$$T^2(\text{spin}) = \tilde{T}^{20} = -\tilde{T}^{02} \quad -(29)$$
$$T^3(\text{spin}) = \tilde{T}^{30} = -\tilde{T}^{03}$$
$$T^1(\text{orb}) = \tilde{T}^{23} = -\tilde{T}^{32}$$
$$T^2(\text{orb}) = \tilde{T}^{31} = -\tilde{T}^{13} \quad -(30)$$
$$T^3(\text{orb}) = \tilde{T}^{12} = -\tilde{T}^{21}$$

From eqs. (7) to (10) and (26) to (30):

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{T}^a(\text{spin}) &= 0 \quad -(31) \\ \frac{1}{c}\frac{\partial \underline{T}^a(\text{spin})}{\partial t} + \underline{\nabla} \times \underline{T}^a(\text{orb}) &= \underline{0} \quad -(32) \end{aligned}}$$

In electromagnetism these become the ECE Gauss law and Faraday law of induction:

6)

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(32)$$

$$\frac{\partial \underline{B}^a}{\partial t} + \underline{\nabla} \times \underline{E}^a = \underline{0} \quad -(33)$$

Therefore in electromagnetism the magnetic flux density is due to spin torsion and the electric field strength is due to orbital torsion. In gravitational theory the gravitomagnetic field is due to spin torsion and the gravitational field is due to orbital torsion.

In addition to the vector equations (31) and (32), the constraint (5) adds two more vector equations as follows. For the case

$$\nu = 0 \quad -(34)$$

$$\omega^a{}_{1b} \tilde{T}^{b10} + \omega^a{}_{2b} \tilde{T}^{b20} + \omega^a{}_{3b} \tilde{T}^{b30} \quad -(35)$$

$$= \tilde{R}^a{}_1{}^{10} + \tilde{R}^a{}_2{}^{20} + \tilde{R}^a{}_3{}^{30}$$

$$= q^b{}_1 \tilde{R}^a{}_b{}^{10} + q^b{}_2 \tilde{R}^a{}_b{}^{20} + q^b{}_3 \tilde{R}^a{}_b{}^{30}$$

i.e.

$$\boxed{\underline{\omega}^a{}_b \cdot \underset{\sim}{T}^b(\text{spin}) = \underline{q}^b \cdot \underset{\sim}{R}^a{}_b(\text{spin})} \quad -(36)$$

in which:

$$\underline{R}^a{}_b(\text{spin}) = R^a{}_b{}^1(\text{spin})\, \underline{i} + R^a{}_b{}^2(\text{spin})\, \underline{j} + R^a{}_b{}^3(\text{spin})\, \underline{k} \quad -(37)$$

7) with

$$R^a{}_b{}^1(\text{spin}) = \tilde{R}^a{}_b{}^{10} \text{ etc} \quad -(38)$$

as in eqns. (29) and (30) for torsion.

For the indices:

$$n = 1, 2, 3 \quad -(39)$$

it follows that:

$$\boxed{\begin{aligned} &\omega^a{}_{0b}\, T^b(\text{spin}) + \underline{\omega}^a{}_b \times \underline{T}^b(\text{orbital}) \\ &= q^b{}_0\, \underline{R}^a{}_b(\text{spin}) + \underline{q}^b \times \underline{R}^a{}_b(\text{orbital}) \end{aligned}} \quad -(40)$$

In these equations:

$$\underline{\omega}^a{}_b = \omega^a_{1b}\,\underline{i} + \omega^a_{2b}\,\underline{j} + \omega^a_{3b}\,\underline{k} \quad -(41)$$

$$\underline{q}^b = q^b_1\,\underline{i} + q^b_2\,\underline{j} + q^b_3\,\underline{k} \quad -(42)$$

and

$$\omega^a_{\mu b} = \left(\omega^a_{0b}, -\underline{\omega}^a{}_b\right) \quad -(43)$$

$$q^b = \left(q^b_0, -\underline{q}^b\right) \quad -(44)$$

The two constraint equations (36) and (40) mean that the charge current density vanishes to give eq. (4).

As in recent notes the Cartan identity (1) can be written as:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b - \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad - (45)$$

and the constraint (5) means:

$$\boxed{\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = 0} \quad - (46)$$

The Cartan identity (1) also implies:

$$\boxed{\frac{1}{c}\frac{\partial T^{a0}}{\partial t} + \omega^a{}_{0b} T^{b0} = R^a{}_0{}^0} \quad - (47)$$

where, for each a and for both spin and orbital cases:

$$T^\mu = (T^0, \underline{T}) \quad - (48)$$

and

$$R^a{}_b{}^\mu = (R^a{}_b{}^0, \underline{R}^a{}_b) \quad - (49)$$

Therefore the Cartan identity and the constraint of vanishing magnetic charge current density produce a set of vector equations which can be used with two more vector equations for torsion from the first Cartan identity and two more equations for curvature from the second Cartan identity.

9) Summary

Eq. (1) gives:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b - \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(50)$$

and

$$\frac{1}{c}\frac{\partial T^{a0}}{\partial t} + \omega^a{}_{0b} T^{b0} = R^a{}_0{}^0 \quad -(51)$$

Eq. (4) gives:

$$\underline{\nabla} \cdot \underset{\sim}{T}^a(\text{spin}) = 0 \quad -(52)$$

$$\frac{1}{c}\frac{\partial \underset{\sim}{T}^a(\text{spin})}{\partial t} + \underline{\nabla} \times \underset{\sim}{T}^a(\text{orb}) = \underline{0} \quad -(53)$$

Eq. (5) gives:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = 0 \quad -(54)$$

$$\underline{\omega}^a{}_b \cdot \underset{\sim}{T}^b(\text{spin}) = \underline{q}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad -(55)$$

$$\omega^a{}_{0b} \underset{\sim}{T}^b(\text{spin}) + \underline{\omega}^a{}_b \times \underset{\sim}{T}^b(\text{orb}) = q_0^b \underline{R}^a{}_b(\text{spin}) + \underline{q}^b \times \underline{R}^a{}_b(\text{orb}) \quad -(56)$$

The first Cartan structure equation gives:

$$\underset{\sim}{T}^a(\text{spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \quad -(57)$$

$$\underset{\sim}{T}^a(\text{orb}) = -\underline{\nabla} q_0^a - \frac{1}{c}\frac{\partial \underline{q}^a}{\partial t} - \omega^a{}_{0b}\,\underline{q}^b + q_0^b\,\underline{\omega}^a{}_b \quad -(58)$$

The second Cartan structure equation gives:

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(59)$$

$$\underline{R}^a{}_b(\text{orb}) = -\underline{\nabla}\, \omega^a{}_{0b} - \frac{1}{c}\frac{\partial \underline{\omega}^a{}_b}{\partial t} - \omega^a{}_{0c}\, \underline{\omega}^c{}_b + \omega^c{}_{0b}\, \underline{\omega}^a{}_c \quad -(60)$$

Therefore there are ten equations available in total, because eq. (50) reduces to eq. (54). There are only two fundamental unknowns, the tetrad $q^a{}_\mu$ and the spin connection $\omega^a{}_{\mu b}$. These equations give many different kinds of spin connection resonance. Finally they are all subjected to antisymmetry constraints.