255(5): Vector Format of the Cartan Identity with Timelike Indices.

In general the Cartan identity is:

$$D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} + D_\nu T^a{}_{\rho\mu} := R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \quad -(1)$$

For spacelike indices:

$$D_1 T^a{}_{23} + D_2 T^a{}_{31} + D_3 T^a{}_{12} := R^a{}_{123} + R^a{}_{231} + R^a{}_{312} \quad -(2)$$

In note 254(7) it was shown that this tensorial identity gives the vector identity:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b - \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(3)$$

If the structure of the homogeneous field equation is assumed then:

$$\underline{\nabla} \cdot \underline{\omega}^b{}_c \times \underline{q}^c = 0 \quad -(4)$$

In note 254(5) it was shown that for timelike indices:

$$D_0 T^a{}_{23} + D_2 T^a{}_{30} + D_3 T^a{}_{02} = R^a{}_{023} + R^a{}_{230} + R^a{}_{302} \quad -(5)$$

$$D_0 T^a{}_{31} + D_1 T^a{}_{03} + D_3 T^a{}_{10} = R^a{}_{031} + R^a{}_{103} + R^a{}_{310} \quad -(6)$$

$$D_0 T^a{}_{12} + D_1 T^a{}_{20} + D_2 T^a{}_{01} = R^a{}_{012} + R^a{}_{120} + R^a{}_{201} \quad -(7)$$

Eqs. (5) to (7) are antisymmetrized tensor products. In order to translate them into vector notation we:

2)

$$T^{a0} = \epsilon^{023} T^{a}{}_{23}$$

$$T^{a2} = \epsilon^{230} T^{a}{}_{30} \qquad (8)$$

$$T^{a3} = \epsilon^{302} T^{a}{}_{02}$$

and so on. The Levi-Civita symbols in eq. (8) must be defined systematically. The four dimensional Levi-Civita symbol is first defined as:

$$\epsilon_{0123} = 1 \qquad (9)$$

The unit vector in four dimensions is defined as:

$$e^{\mu} = (e^{0}, e^{1}, e^{2}, e^{3}) = (1, 1, 1, 1) \qquad (10)$$

Then

$$\epsilon_{123} = \epsilon_{0123}\, e^{0} = 1$$

$$\epsilon_{132} = \epsilon_{0132}\, e^{0} = -1 \qquad (11)$$

$$\epsilon_{312} = \epsilon_{0312}\, e^{0} = 1$$

etc.

The three dimensional Levi-Civita symbols with a zero index are defined similarly:

$$\epsilon_{012} = \epsilon_{0123}\, e^{3} = 1$$

$$\epsilon_{021} = \epsilon_{0213}\, e^{3} = -1 \qquad (12)$$

$$\epsilon_{201} = \epsilon_{2013}\, e^{3} = 1$$

etc.

Also:

$$\epsilon_{023} = \epsilon_{0123}\, e^{1} = 1$$

$$\epsilon_{032} = \epsilon_{0132}\, e^{1} = -1 \qquad (13)$$

$$\epsilon_{[illegible]} = \epsilon_{[illegible]}\, e^{1} = 1$$

3) $\epsilon_{013} = \epsilon_{0123}\, e^2 = 1$

$\epsilon_{031} = \epsilon_{0321}\, e^2 = -1$ — (14)

$\epsilon_{301} = \epsilon_{3021}\, e^2 = 1$

etc.

Therefore:

$$\epsilon_{023} = \epsilon_{302} = \epsilon_{230} = 1 \quad -(15)$$

$$\epsilon_{012} = \epsilon_{201} = \epsilon_{120} = 1 \quad -(16)$$

$$\epsilon_{013} = \epsilon_{301} = \epsilon_{130} = 1 \quad -(17)$$

and $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$ — (18)

In order to raise indices in eqs. (15) to (18) note that:

$$\epsilon^{\mu\nu\rho\sigma} = g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g^{\sigma\delta} \epsilon_{\alpha\beta\gamma\delta} \quad -(19)$$

For the Minkowski metric:

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \quad -(20)$$

So $\epsilon^{0123} = -\epsilon_{0123} = -1$. — (21)

The covariant unit vector is:

$$e_\mu = (1, -1, -1, -1) \quad -(22)$$

So in Minkowski spacetime:

4) $$\epsilon^{123} = \epsilon^{0123}\epsilon_0 = -1 \quad -(23)$$

So:
$$\epsilon^{123} = \epsilon^{312} = \epsilon^{231} = -1. \quad -(24)$$

Similarly:
$$\epsilon^{023} = \epsilon^{302} = \epsilon^{230} = 1 \quad -(25)$$
$$\epsilon^{012} = \epsilon^{201} = \epsilon^{120} = 1 \quad -(26)$$
$$\epsilon^{013} = \epsilon^{301} = \epsilon^{130} = 1 \quad -(27)$$

Therefore eq. (5) becomes:

$$D_0 T^{a0} + D_2 T^{a2} + D_3 T^{a3} := R^{a}{}_{0}{}^{0} + R^{a}{}_{2}{}^{2} + R^{a}{}_{3}{}^{3}. \quad -(28)$$

$$= q^{b}{}_{0} R^{a}{}_{b}{}^{0} + q^{b}{}_{2} R^{a}{}_{b}{}^{2} + q^{b}{}_{3} R^{a}{}_{b}{}^{3}$$

Similarly eqs (6) and (7) become:

$$D_0 T^{a0} + D_1 T^{a1} + D_3 T^{a3} = q^{b}{}_{0} R^{a}{}_{b}{}^{0} + q^{b}{}_{2} R^{a}{}_{b}{}^{2} + q^{b}{}_{3} R^{a}{}_{b}{}^{3} \quad -(29)$$

nd:

$$T^{a0} + D_1 T^{a1} + D_2 T^{a2} = q^{b}{}_{0} R^{a}{}_{b}{}^{0} + q^{b}{}_{1} R^{a}{}_{b}{}^{1} + q^{b}{}_{2} R^{a}{}_{b}{}^{2} \quad -(30)$$

Adding eqs. (28) to (30) gives

5)

$$3D_0 T^{a0} + 2\left(D_1 T^{a1} + D_2 T^{a2} + D_3 T^{a3}\right)$$
$$= 3q^b{}_0 R^a{}_b{}^0 + 2\left(q^b{}_1 R^a{}_b{}^1 + q^b{}_2 R^a{}_b{}^2 + q^b{}_3 R^a{}_b{}^3\right) \quad -(31)$$

However, eq. (2) gives:

$$D_1 T^{a1} + D_2 T^{a2} + D_3 T^{a3} = q^b{}_1 R^a{}_b{}^1 + q^b{}_2 R^a{}_b{}^2 + q^b{}_3 R^a{}_b{}^3 \quad -(32)$$

which in vector notation is eq. (3).

Therefore eq. (31) gives:

$$\boxed{D_0 T^{a0} = q^b{}_0 R^a{}_b{}^0 = R^a{}_0{}^0} \quad -(33)$$

i.e.

$$\partial_0 T^{a0} + \omega^a{}_{0b} T^{b0} = R^a{}_0{}^0 \quad -(34)$$

or

$$\boxed{\frac{1}{c}\frac{\partial T^{a0}}{\partial t} + \omega^a{}_{0b} T^{b0} = R^a{}_0{}^0} \quad -(35)$$

This equation gives a new structure for spin connection resonance.