152(2): Development of the Term Quadratic in the Potential.

This is described by the Hamiltonian:

$$\hat{H}\psi = \frac{e^2}{2m} \underline{\sigma} \cdot \underline{A}\, \underline{\sigma} \cdot \underline{A}\, \psi \quad -(1)$$

Now note that:

$$\underline{\sigma} \cdot \underline{A} = \frac{\underline{\sigma} \cdot \underline{r}}{r^2} \left(\underline{r} \cdot \underline{A} + i \underline{\sigma} \cdot \underline{r} \times \underline{A} \right) \quad -(2)$$

$$\left(\underline{\sigma} \cdot \underline{A}\right)^* = \frac{\underline{\sigma} \cdot \underline{r}}{r^2} \left(\underline{r} \cdot \underline{A} - i \underline{\sigma} \cdot \underline{r} \times \underline{A} \right) \quad -(3)$$

For a uniform magnetic field:

$$\underline{A} = \frac{1}{2} \underline{B} \times \underline{r} \quad -(4)$$

and

$$\underline{r} \cdot \underline{A} = 0 \quad -(5)$$

So:

$$\left(\underline{\sigma} \cdot \underline{A}\right)\left(\underline{\sigma} \cdot \underline{A}\right)^* = \frac{1}{r^2} \underline{\sigma} \cdot \underline{r} \times \underline{A}\, \underline{\sigma} \cdot \underline{r} \times \underline{A}$$

$$= \frac{1}{r^2} \left((\underline{r} \cdot \underline{r})(\underline{A} \cdot \underline{A}) - (\underline{r} \cdot \underline{A})(\underline{A} \cdot \underline{r}) \right)$$

$$= A^2 \quad -(6)$$

Therefore the Hamiltonian (1) can be developed

2) as:

$$\hat{H}\phi = \frac{e^2}{2m} \underline{\sigma} \cdot \underline{A}\, \underline{\sigma} \cdot \underline{A}\phi$$

$$= \frac{e^2}{2m} \underline{\sigma} \cdot \underline{A} \left(\underline{\sigma} \cdot \underline{A}\right)^* \phi \qquad -(7)$$

$$= \frac{e^2 A^2}{2m} \phi$$

for real $\underline{A}$.

From eqs. (4) and (6):

$$\frac{1}{r^2} \underset{\sim}{r} \times \underline{A} \cdot \underline{r} \times \underline{A} = \frac{1}{4r^2} \underset{\sim}{r} \times \left(\underline{B} \times \underline{r}\right) \cdot \underline{r} \times \left(\underline{B} \times \underline{r}\right).$$

$$= \frac{1}{4r^2} \left(r^2 \underline{B} \times \underline{r} \cdot \underline{B} \times \underline{r} - \underline{r} \cdot \underline{A}\, \underline{A} \cdot \underline{r} \right) \qquad -(8)$$

$$= \frac{1}{4} \underline{B} \times \underline{r} \cdot \underline{B} \times \underline{r}$$

$$= \frac{1}{4} \left(B^2 r^2 - \left(\underset{\sim}{r} \cdot \underline{B}\right)\left(\underset{\sim}{r} \cdot \underline{B}\right)\right)$$

Therefore:

$$\boxed{\hat{H}\phi = \frac{e^2}{8m} \left(B^2 r^2 - \left(\underline{r} \cdot \underline{B}\right)\left(\underline{r} \cdot \underline{B}\right)\right)\phi \qquad -(9)}$$

which is a hamiltonian quadratic in the magnetic flux density.

If: $$\underline{B} = B_z \underline{k} \quad -(10)$$

then:

$$\hat{H}\phi = \frac{e^2 B_z^2}{8m}\left(r^2 - Z^2\right)\phi \quad -(11)$$

$$= \frac{e^2 B_z^2}{8m} r^2 \left(1 - \cos^2\theta\right)\phi$$

in spherical polar coordinates

The expectation value of the energy are:

$$E = \frac{e^2 B_z^2}{8m} \int \phi^* r^2 \left(1 - \cos^2\theta\right) \phi \, d\tau$$

$$-(12)$$

where

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi. \quad -(13)$$

Therefore:

$$E = \frac{e^2 B_z^2}{8m} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \phi^* r^4 \left(1 - \cos^2\theta\right) \sin\theta \, \phi \, dr \, d\theta \, d\phi$$

$$-(14)$$

The <u>magnetizability</u> is:

$$\chi = \frac{e^2}{8m} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \psi^* r^4 \left(1 - \cos^2\theta\right) \sin\theta \, \psi \, dr \, d\theta \, d\phi \qquad -(15)$$

and course worked out for the hydrogenic wave functions.

So:

$$E = \chi B^2 \qquad -(15)$$

is the second order perturbation energy.
