252(10): Summary of Types of Hamiltonian

The remaining types of Hamiltonian to be calculated are as follows. The first is:

$$\hat{H}\phi = \frac{ie}{4m^2c^2}\,\underline{\sigma}\cdot\underline{L}\left(\frac{\phi}{r^2}\,\underline{r}\cdot\underline{p}\,\phi\right) \quad -(1)$$

with $\underline{r}\cdot\underline{p}$ regarded as the function:

$$\underline{r}\cdot\underline{p} = \underline{\sigma}\cdot\underline{r}\,\underline{\sigma}\cdot\underline{p} - i\,\underline{\sigma}\cdot\underline{L} \quad -(2)$$

so:

$$\text{Real}(\hat{H}\phi) = \frac{e}{4m^2c^2}\,\underline{\sigma}\cdot\underline{L}\left(\frac{\phi}{r^2}\,\underline{\sigma}\cdot\underline{L}\,\phi\right) \quad -(3)$$

which is the same as $\hat{H}_4$ of note 252(6) except for a sign change.

The second is:

$$\hat{H}\phi = \frac{ie}{4m^2c^2}\,\underline{r}\cdot\underline{p}\left(\frac{\phi}{r^2}\,\underline{\sigma}\cdot\underline{L}\,\phi\right) \quad -(4)$$

with $\underline{r}\cdot\underline{p}$ regarded as the function (2). This gives the same result as eq. (3).

Summary

1)

$$\hat{H}_1\phi = \frac{e}{4m^2c^2}\,\underline{r}\cdot\underline{p}\left(\frac{\phi}{r^2}\,\underline{r}\cdot\underline{p}\,\phi\right) \quad -(5)$$

a) Both $\underline{r}\cdot\underline{p}$ as operators (note 252(9)):

2) $$\hat{H}_1 \phi = -\frac{e^2 \hbar^2}{16\pi\epsilon_0 m^2 c^2}\left(\frac{2}{r}\frac{d\phi}{dr} - \frac{1}{r}\frac{d^2\phi}{dr^2}\right) \quad -(6)$$

b) First $\underline{r}\cdot\underline{p}$ as function and second $\underline{r}\cdot\underline{p}$ as operator (note 255(4)):

$$\hat{H}_1 \phi = \frac{e^2 \hbar}{16\pi\epsilon_0 m^2 c^2 r^3}\,\underline{\sigma}\cdot\underline{L}\left(3\phi - r\frac{d\phi}{dr}\right)$$

$$= \frac{e^2 \hbar^2}{16\pi\epsilon_0 m^2 c^2 r^3}\left(j(j+1) - l(l+1) - s(s+1)\right)\left(3\phi - r\frac{d\phi}{dr}\right) \quad -(7)$$

2)a) $$\hat{H}_2 \phi = -\frac{e}{4m^2c^2}\frac{\phi}{r^2}\,\underline{\sigma}\cdot\underline{L}\,\underline{\sigma}\cdot\underline{L}\,\phi \quad -(8)$$

$$= \frac{e^2 \hbar^2}{16\pi\epsilon_0 m^2 c^2 r^3}\left(2l(l+1) - j(j+1) + s(s+1)\right)\phi$$

b) $$\hat{H}_2 \phi = \frac{ie}{4m^2c^2}\,\underline{\sigma}\cdot\underline{L}\left(\frac{\phi}{r^2}\,\underline{r}\cdot\underline{p}\,\phi\right), \ \underline{r}\cdot\underline{p} \text{ operator}$$

$$= \frac{e}{4m^2c^2}\,\underline{\sigma}\cdot\underline{L}\left(\frac{\phi}{r^2}\,\underline{\sigma}\cdot\underline{L}\,\phi\right)$$

$$= \frac{-e^2 \hbar^2}{16\pi\epsilon_0 m^2 c^2 r^3}\left(2l(l+1) - j(j+1) + s(s+1)\right)\phi \quad -(9)$$

c) $\hat{H}_2 \psi = \frac{ie}{4m^2c^2} \underline{r} \cdot \underline{p} \left(\frac{\phi}{r^2} \underline{\sigma} \cdot \underline{L} \psi \right)$, $\underline{r} \cdot \underline{p}$ operator

$$= \frac{e}{4m^2c^2} \underline{\sigma} \cdot \underline{L} \left(\frac{\phi}{r^2} \underline{\sigma} \cdot \underline{L} \psi \right)$$

$$= -\frac{e^2 \hbar^2}{16\pi \epsilon_0 m^2 c^2 r^3} \left(2l(l+1) - j(j+1) + s(s+1) \right) \psi \quad -(10)$$

3) $\hat{H}_3 \psi = i \frac{e}{4m^2c^2} \underline{r} \cdot \underline{p} \left(\frac{\phi}{r^2} \underline{\sigma} \cdot \underline{L} \psi \right)$, $-(11)$

$\underline{r} \cdot \underline{p}$ operator

$$= \frac{3e^2 \hbar^2}{16\pi \epsilon_0 m^2 c^2 r^3} \left(j(j+1) - l(l+1) - s(s+1) \right) \psi$$

There are only three distinct types of hamiltonian as follows:

Type I

$$\boxed{\hat{H} \psi = -\frac{e^2 \hbar^2}{16\pi \epsilon_0 m^2 c^2} \left(\frac{2}{r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \right)}$$

and two other types:

4)

Type II

$$\boxed{\hat{H}\phi = \frac{e^2\hbar^2}{16\pi\epsilon_0 m^2c^2r^3}\left(j(j+1)-\ell(\ell+1)-s(s+1)\right)\left(3\phi - r\frac{\partial\phi}{\partial r}\right)}$$

Type III

$$\boxed{\hat{H}\phi = \frac{e^2\hbar^2}{16\pi\epsilon_0 m^2c^2r^3}\left(2\ell(\ell+1)-j(j+1)+s(s+1)\right)\phi}$$