252(4): Development of the Conventional Spin - Orbit Term of the Fermion Equation

The conventional spin orbit term is developed from the conventional hamiltonian:

$$\hat{H}\phi = \frac{c^2 \underline{\sigma} \cdot \underline{p} \, \underline{\sigma} \cdot \underline{p}}{E - e\phi + mc^2} \phi \quad - (1)$$

with:

$$\phi = - \frac{e}{4\pi \epsilon_0 r} \quad - (2)$$

in the hydrogen atom. In the approximation:

$$E \sim mc^2 \quad - (3)$$

eq. (1) becomes:

$$\hat{H}\phi = \frac{1}{2m} \underline{\sigma} \cdot \underline{p} \left(1 + \frac{e\phi}{2mc^2} \right) \underline{\sigma} \cdot \underline{p} \, \phi \quad - (4)$$

and the spin orbit hamiltonian is:

$$\hat{H}_1 \phi = \frac{e}{4m^2c^2} \underline{\sigma} \cdot \underline{p} \, \phi \, \underline{\sigma} \cdot \underline{p} \, \phi \quad - (5)$$

Now use:

$$\underline{\sigma} \cdot \underline{p} = \frac{\underline{\sigma} \cdot \underline{r}}{r^2} \left(\underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L} \right) - (6)$$

from Pauli algebra.

2) Then:

$$\hat{H}_1 \phi = \frac{e}{4m^2c^2}\left(\underset{\sim}{r} \cdot \underline{p} + i\underline{\sigma} \cdot \underline{L}\right) \frac{\phi}{r^2} \left(\underline{r} \cdot \underline{p} + i\underline{\sigma} \cdot \underline{L}\right) \phi \quad -(7)$$

<u>which</u> <u>gives</u> <u>several</u> new <u>effects</u>.

The first of these is:

$$\hat{H}_{11} \phi = \frac{e}{4m^2c^2} \underline{r} \cdot \underline{p} \left(\frac{\phi}{r^2} \underline{r} \cdot \underline{p} \phi \right) \quad -(8)$$

where

$$\underset{\sim}{r} \cdot \underline{p} \phi = -i\hbar r \frac{\partial \phi}{\partial r} \quad -(9)$$

So:

$$\hat{H}_{11} \phi = -\frac{ie\hbar r}{4m^2c^2} \frac{\partial}{\partial r} \left(\frac{\phi}{r^2} \underline{r} \cdot \underline{p} \phi \right) \quad -(10)$$

Now use:

$$\begin{aligned} \underline{\sigma} \cdot \underline{r}\, \underline{\sigma} \cdot \underline{p} &= \underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{r} \times \underline{p} \\ &= \underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L} \quad -(11) \end{aligned}$$

So

$$\underline{r} \cdot \underline{p} = \underline{\sigma} \cdot \underline{r}\, \underline{\sigma} \cdot \underline{p} - i \underline{\sigma} \cdot \underline{L} \quad -(12)$$

Therefore the real part of $\hat{H}_{11}$ in eq. (10)

is:

3)

$$\text{Real}\left(\hat{H}_{11}\psi\right) = -\frac{e\hbar r}{4m^2c^2}\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\underline{\sigma}\cdot\underline{L}\psi\right) \quad -(13)$$

$$= -\frac{e\hbar r}{4m^2c^2}\underline{\sigma}\cdot\underline{L}\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\psi\right)$$

$$= -\frac{e\hbar r}{4m^2c^2}\underline{\sigma}\cdot\underline{L}\left[\left(\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\right)\right)\psi + \frac{\phi}{r^2}\frac{\partial\psi}{\partial r}\right]$$

where

$$\phi = -\frac{e}{4\pi r\epsilon_0} \quad -(14)$$

Therefore:

$$\frac{\partial}{\partial r}\left(\frac{\phi}{r^2}\right) = \frac{3e}{4\pi\epsilon_0 r^4} \quad -(15)$$

So:

$$\boxed{\hat{H}_{11}\psi = \frac{e^2\hbar}{16m^2c^2\epsilon_0 r^3}\underline{\sigma}\cdot\underline{L}\left(3\psi - r\frac{\partial\psi}{\partial r}\right)} \quad -(16)$$

This gives energy expectation values:

$$E_1 = \frac{3e^2\hbar\,\underline{\sigma}\cdot\underline{L}}{16m^2c^2\epsilon_0}\int\frac{\psi^*\psi}{r^3}\,d\tau \quad -(17)$$

and

$$E_2 = \frac{-e^2\hbar\,\underline{\sigma}\cdot\underline{L}}{16m^2c^2\epsilon_0}\int\frac{\psi^*}{r^2}\frac{\partial\psi}{\partial r}\,d\tau \quad -(18)$$

which can be computed for the H wave functions.