252(3) : Application to Particle Collision Theory

Consider the collision process:

$$E + E_1 = E' + E'' \quad - (1)$$

$$\underline{p} + \underline{p}_1 = \underline{p}' + \underline{p}'' \quad - (2)$$

In analogy with previous notes: let:

$$\underline{p}_1 = e\underline{A} \,, \; E_1 = e\phi \quad - (3)$$

The Hamiltonian considered in 252(2) is:

$$\hat{H}\phi = \frac{1}{2m} \underline{\sigma} \cdot \underline{p}_1 \underline{\sigma} \cdot \underline{p}_1 \phi \quad - (4)$$

where ϕ is the wavefunction of the particle with momentum $\underline{p}$.

Now let:

$$\underline{p}_1 = \frac{1}{2} \underline{\beta} \times \underline{r} \quad - (5)$$

in analogy with $\underline{A} = \frac{1}{2} \underline{B} \times \underline{r} \quad - (6)$

for a uniform magnetic field. The angular momentum associated with p_1 is:

$$\underline{L}_1 = \underline{r} \times \underline{p}_1 = \frac{1}{2} \underline{r} \times (\underline{\beta} \times \underline{r})$$

$$= \frac{1}{2} \left(r^2 \underline{\beta} - \underline{r} (\underline{r} \cdot \underline{\beta}) \right) \quad - (7)$$

Note that:

$$\underline{r} \cdot \underline{p}_1 = \frac{1}{2} \underline{r} \cdot \underline{\beta} \times \underline{r}$$

$$= \frac{1}{2} \underline{\beta} \cdot \underline{r} \times \underline{r} = 0 \quad - (8)$$

2) So $\underline{r}$ is perpendicular to $\underline{p}_1$. The Z axis angular momentum in spherical polar coordinates is:

$$L_{1Z} = \frac{1}{2} r^2 \beta_Z (1 - \cos^2\theta) \quad -(9)$$

so

$$\boxed{\beta_Z = m_1 \frac{d\theta}{dt} = m_1 \omega} \quad -(10)$$

where m_1 is the mass of the particle with momentum $\underline{p}_1$. Here:

$$\omega = \frac{d\theta}{dt} \quad -(11)$$

is the angular velocity of particle 1, the particle with momentum $\underline{p}_1$.

From eq. (4), and in analogy with note 252(2):

$$\hat{H}\phi = \frac{1}{8m}\left(\beta^2 r^2 - (\underline{r}\cdot\underline{\beta})(\underline{r}\cdot\underline{\beta})\right)\phi \quad -(12)$$

and if

$$\underline{\beta} = \beta_Z \underline{k} \quad -(13)$$

$$\boxed{\begin{aligned}\hat{H}\phi &= \frac{\beta_Z^2}{8m} r^2 (1 - \cos^2\theta)\phi \\ &= \frac{1}{8}\frac{m_1^2 \omega^2 r^2}{m}(1 - \cos^2\theta)\phi\end{aligned}} \quad -(14)$$

3) The energy expectation values from eq. (14) are:

$$E = \frac{m_1^2 \omega^2}{8m} \int \phi^* \left(1 - \cos^2\theta\right) r^2 \phi \, d\tau \quad - (15)$$

If ϕ are the hydrogenic wave functions then E are the energy levels of an H atom interacting with a particle of momentum p_1 and angular momentum $\underline{L_1}$ given by eq. (7).