252(8): Evaluation of the $\hat{H}_3$ Hamiltonian.

The $\hat{H}_3$ Hamiltonian is defined by:

$$\hat{H}_3 \phi = \frac{i e}{4 m^2 c^2} \underline{r} \cdot \underline{p} \left(\frac{\phi}{r^2} \underline{\sigma} \cdot \underline{L} \phi \right) \quad -(1)$$

where:

$$\underline{r} \cdot \underline{p} \, \phi = -i\hbar r \frac{\partial \phi}{\partial r} \quad -(2)$$

Therefore:

$$\hat{H}_3 \phi = \frac{e \hbar r}{4 m^2 c^2} \frac{\partial}{\partial r} \left(\frac{\phi}{r^2} \underline{\sigma} \cdot \underline{L} \phi \right)$$

$$= \frac{e \hbar r}{4 m^2 c^2} \left(\left(\frac{\partial}{\partial r} \left(\frac{\phi}{r^2} \right) \right) \underline{\sigma} \cdot \underline{L} \phi + \frac{\phi}{r^2} \frac{\partial}{\partial r} \left(\underline{\sigma} \cdot \underline{L} \phi \right) \right)$$

where

$$\phi = - \frac{e}{4 \pi \epsilon_0 r} \quad -(3)$$

Therefore:

$$\hat{H}_3 \phi = - \frac{e^2 \hbar r}{16 \pi \epsilon_0 m^2 c^2} \frac{\partial}{\partial r} \left(\frac{1}{r^3} \right) \left(j(j+1) - l(l+1) - s(s+1) \right) \phi$$

$$= \frac{3 e^2 \hbar}{16 \pi \epsilon_0 m^2 c^2 r^3} \left(j(j+1) - l(l+1) - s(s+1) \right) \phi \quad -(4)$$

This is the same as E_1 of note 252(4), eq. (17).