250(3): Usual Derivation of the Anomalous Zeeman Effect

This is a nineteen twenties derivation first given empirically by Landé. The hamiltonian is written as:

$$H = -\frac{e}{2m}\left(\underline{L} + 2\underline{S}\right)\cdot\underline{B} \quad -(1)$$

where $\underline{L}$ is the orbital angular momentum and $\underline{S}$ the spin angular momentum. Here $-e$ and m are the charge and mass of the electron. If the applied magnetic flux density $\underline{B}$ is in the Z axis then:

$$H = -\frac{e}{2m}\left(L_Z + 2S_Z\right)B_Z \quad -(2)$$

However, only the total angular momentum is conserved and only the Z component of $\underline{J}$ is well defined:

$$\hat{J}_Z\,\phi = m_J\,\hbar\,\phi \quad -(3)$$

The usual textbook derivation is rather vague and confusing, so full details are given here. The usual derivation relies on the assertions:

$$\underline{L}\cdot\underline{B} = \frac{1}{J^2}\,\underline{L}\cdot\underline{J}\;\underline{J}\cdot\underline{B} \quad -(4)$$

$$\underline{S}\cdot\underline{B} = \frac{1}{J^2}\,\underline{S}\cdot\underline{J}\;\underline{J}\cdot\underline{B} \quad -(5)$$

However,
$$\underline{L}\cdot\underline{B} = L_X B_X + L_Y B_Y + L_Z B_Z \quad -(6)$$

but:

$$\underline{L}\cdot\underline{J}\;\underline{J}\cdot\underline{B} = \frac{1}{J^2}\left(L_X J_X + L_Y J_Y + L_Z J_Z\right)\left(J_X B_X + J_Y B_Y + J_Z B_Z\right) \quad -(7)$$

2) where

$$J^2 = J_x^2 + J_y^2 + J_z^2 \quad -(8)$$

So eq. (4) is true if and only if:

$$\underline{J} = J_z \underline{k}, \quad \underline{B} = B_z \underline{k} \quad -(9)$$

Similarly for eq. (5). So the usual theory of the anomalous Zeeman effect is only an approximation.

Now use:

$$\underline{J} = \underline{L} + \underline{S} \quad -(10)$$

So

$$\begin{aligned} J^2 &= L^2 + S^2 + 2\underline{L}\cdot\underline{S} \\ &= L^2 + S^2 + 2(\underline{J} - \underline{S})\cdot\underline{S} \\ &= L^2 - S^2 + 2\underline{J}\cdot\underline{S} \end{aligned} \quad -(11)$$

Therefore:

$$\underline{S}\cdot\underline{J} = \frac{1}{2}\left(J^2 + S^2 - L^2\right) \quad -(12)$$

Similarly:

$$J^2 = L^2 + S^2 + 2\underline{L}\cdot(\underline{J} - \underline{L}) \quad -(13)$$

So

$$\underline{L}\cdot\underline{J} = \frac{1}{2}\left(J^2 + L^2 - S^2\right) \quad -(14)$$

Therefore the Hamiltonian (7) is:

$$H = -\frac{e}{2m}\cdot\frac{1}{2J^2}\left(\underline{L}\cdot\underline{J} + 2\underline{S}\cdot\underline{J}\right)\underline{J}\cdot\underline{B} \quad -(15)$$

3) It has been rewritten in terms of a hamiltonian in the total angular momentum $\underline{J}$.

Therefore:

$$H = -\frac{e}{2m}\cdot\frac{1}{2J}\left(J^2+L^2-S^2+2\left(J^2+S^2-L^2\right)\right)\underline{J}\cdot\underline{B}$$

$$= -\frac{e}{2m}\left(1+\frac{J^2+S^2-L^2}{2J^2}\right)\underline{J}\cdot\underline{B}$$

$$= -\frac{e}{2m}g_L\,\underline{J}\cdot\underline{B} \qquad -(16)$$

where the Landé factor is defined by:

$$g_L = 1+\frac{J^2+S^2-L^2}{2J^2} \qquad -(17)$$

Finally use:

$$\hat{J}^2\psi = \hbar^2 J(J+1)\psi \qquad -(18)$$

$$\hat{L}^2\psi = \hbar^2 L(L+1)\psi \qquad -(19)$$

$$\hat{S}^2\psi = \hbar^2 S(S+1)\psi \qquad -(20)$$

to obtain:

$$g_L = 1+\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)} \qquad -(21)$$

It is seen that the well known hamiltonian (16) of the anomalous Zeeman effect is

4) fundamentally different from the ESOR hamiltonian:

$$H_2 = -\frac{e}{2m} \underline{L} \cdot \underline{B} = \frac{e}{2m} \underline{\sigma} \cdot \underline{L} \, \underline{\sigma} \cdot \underline{B} + \dots \quad -(22)$$

which consists only of the orbital angular momentum rewritten using Pauli algebra as:

$$H_2 = -\frac{e}{2m} \underline{L} \cdot \underline{B} = \frac{e}{2m} \underline{\sigma} \cdot \underline{L} \left(\underline{\sigma} \cdot \underline{B} - \frac{1}{r^2} \underline{\sigma} \cdot \underline{r} \, \underline{B} \cdot \underline{r} \right) \quad -(23)$$

The starting hamiltonian of the anomalous Zeeman effect is eq. (1):

$$H = -\frac{e}{2m} \left(\underline{L} + 2 \underline{S} \right) \cdot \underline{B} \quad -(24)$$

The starting hamiltonian of ESOR is:

$$H_2 = -\frac{e}{2m} \underline{L} \cdot \underline{B} \quad -(25)$$

which is rewritten as eq. (23). The hamiltonian (25) is that of the normal Zeeman effect. From eq. (25):

$$\hat{H}_2 \psi = -\frac{e m_L \hbar}{2m} B_Z \quad -(26)$$

because:

$$\hat{L}_Z \psi = m_L \hbar \psi \quad -(27)$$

$$m_L = -L, \dots L \quad -(28)$$