248(9): Derivation of the Darwin Term from the Fermion Equation

Start from the $\hat{H}_{22}$ hamiltonian:

$$\hat{H}_{22}\phi = \frac{e}{4m^2c^2}\left(\underline{\sigma}\cdot(\underline{p}-e\underline{A})\,\phi\,\underline{\sigma}\cdot(\underline{p}-e\underline{A})\right)\phi \quad -(1)$$

And assume that

$$\underline{A} = \underline{0}, \quad -(2)$$

so
$$\hat{H}_{22}\phi = \frac{e}{4m^2c^2}\left(\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p}\right)\phi \quad -(3)$$

In the derivation of spin orbit coupling and the Thomas factor, the first $\underline{p}$ is regarded as an operator and the second p is regarded as a function. In the derivation of the Darwin term both $\underline{p}$'s are regarded as operators.

The $\underline{p}$ operator is defined by:

$$\hat{\underline{p}} = -i\hbar\underline{\nabla} \quad -(4)$$

so
$$-i\hbar\underline{\nabla}\phi = \underline{p}\phi \quad -(5)$$

and the function $\underline{p}$ is the expectation value of the eigenvalue $\underline{p}$ in eq. (5)

$$\underline{p} = \langle\underline{p}\rangle = \int \phi^*\underline{p}\,\phi\,d\tau \quad -(6)$$

Therefore the Darwin term is obtained from:

$$\hat{H}_{22}\phi = \frac{e}{4m^2c^2}\left(\underline{\sigma}\cdot(-i\hbar\underline{\nabla})\,\phi\,\underline{\sigma}\cdot(-i\hbar\underline{\nabla})\right)\phi \quad -(7)$$

2) and is a purely quantum mechanical phenomenon with no classical counterpart.

From eq. (7):

$$\hat{H}_{22}\phi = \frac{-e\hbar^2}{4m^2c^2}\left(\underline{\sigma}\cdot\underline{\nabla}\,\phi\,\underline{\sigma}\cdot\underline{\nabla}\right)\phi. \quad -(8)$$

The first del operator $\underline{\nabla}$ operates on all that follows it, so:

$$\hat{H}_{22}\phi = -\frac{e\hbar^2}{4m^2c^2}\left(\underline{\sigma}\cdot\underline{\nabla}\left(\phi\,\underline{\sigma}\cdot\underline{\nabla}\,\phi\right)\right) \quad -(9)$$

The Leibnitz theorem is used as follows:

$$\underline{\nabla}\left(\phi\,\underline{\sigma}\cdot\underline{\nabla}\,\phi\right) = \left(\underline{\nabla}\,\phi\right)\left(\underline{\sigma}\cdot\underline{\nabla}\,\phi\right) + \phi\,\underline{\nabla}\left(\underline{\sigma}\cdot\underline{\nabla}\,\phi\right) \quad -(10)$$

Therefore:

$$\hat{H}_{22}\phi = \frac{-e\hbar^2}{4m^2c^2}\Big[\underline{\sigma}\cdot\underline{\nabla}\,\phi\,\underline{\sigma}\cdot\underline{\nabla}\,\phi + \underline{\sigma}\cdot\phi\,\underline{\nabla}\left(\underline{\sigma}\cdot\underline{\nabla}\,\phi\right)\Big] \quad -(11)$$

The Darwin term is:

$$\boxed{\hat{H}_{\text{Darwin}}\phi = -\frac{e\hbar^2}{4m^2c^2}\,\underline{\sigma}\cdot\underline{\nabla}\,\phi\,\underline{\sigma}\cdot\underline{\nabla}\,\phi} \quad -(12)$$

The second term in eq. (11) can be developed using:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi\right) = \left(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\psi \quad -(13)$$

so:

$$\hat{H}_{22}\psi = -\frac{e\hbar^2}{4m^2c^2}\left[\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi + \phi\left(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\psi\right] \quad -(14)$$

$$= -\frac{e\hbar^2}{4m^2c^2}\left[\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi + \phi\,\nabla^2\psi\right]$$

The second term is therefore an addition to the usual kinetic energy term:

$$\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}\psi = -\hbar^2\nabla^2\psi \quad -(15)$$

The Darwin term has also been observed experimentally with accuracy.

Finally, in some treatments of the Dirac equation, the positive minimal prescription is used, so that there appears a term $E - mc^2 + e\phi$, which is approximated by:

$$E - mc^2 + e\phi = E' - V \sim T = \frac{p^2}{2m} \quad -(16)$$

In the next note this formalism will be developed in particle collision theory and LENR.