243(7): Number Density of Photons and Mean Square Photon Mass.

Consider the equation defining the density of states and mean energy in the Planck distribution:

$$\rho(\omega) = \frac{dU}{d\omega}, \quad dU = \langle E \rangle dN \quad -(1)$$

as in previous notes. It follows that:

$$dU = \rho(\omega)\, d\omega = \hbar\omega\, dN \quad -(2)$$

and

$$dN = \frac{\rho(\omega)}{\hbar\omega}\, d\omega \quad -(3)$$

for one photon, in which:

$$\langle E \rangle = \hbar\omega. \quad -(4)$$

The density of states is:

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3}\left(\frac{\omega^3}{e^{\hbar\omega/kT} - 1}\right) \quad -(5)$$

so:

$$dN = \frac{1}{\pi^2 c^3}\left(\frac{\omega^2}{e^{\hbar\omega/kT} - 1}\right) \quad -(6)$$

and

$$N = \frac{1}{\pi^2 c^3}\int_0^\infty \left(\frac{\omega^2}{e^{\hbar\omega/kT} - 1}\right) d\omega \quad -(7)$$

is the number of photons per unit volume (i.e.

2) The number of photons per cubic metre of black body radiation).

The mean square photon mass is, from the previous note:

$$\langle m^2 \rangle = \frac{\hbar^2}{c^4} \left(\frac{\omega^2}{e^{\hbar\omega / kT} - 1} \right) \quad -(8)$$

so

$$\boxed{N = \frac{c}{\pi^2 \hbar^2} \int_0^\infty \langle m^2 \rangle \, d\omega} \quad -(9)$$

The integral in eq. (7) is:

$$N = \frac{c}{\pi^2 \hbar^2} \int_0^\infty \langle m^2 \rangle \, d\omega = \left(\frac{2 \zeta(3) k^3}{\pi^2 c^3 \hbar^3} \right) T^3 \quad -(10)$$

where

$$\zeta(3) = 1.20206 \, . \quad -(11)$$

so

$$\boxed{\int_0^\infty \langle m^2 \rangle \, d\omega = \left(\frac{2 \zeta(3) k^3}{c^4 \hbar} \right) T^3} \quad -(11)$$

The integrated mean square photon mass can be worked out from eq. (11).

) Here:

$$k = 1.38066 \times 10^{-23}\, JK^{-1} \quad -(12)$$

$$\hbar = 1.05459 \times 10^{-34}\, Js \quad -(13)$$

$$c = 2.997925 \times 10^{8}\, ms^{-1} \quad -(14)$$

So:

$$\boxed{\int_0^{\infty} \langle m^2 \rangle d\omega \sim 7.4 \times 10^{-69}\, T^3} \quad -(15)$$

From eq. (9), the number of photons per cubic metre of black body radiation is:

$$N \sim 10^{76} \int_0^{\infty} \langle m^2 \rangle d\omega \quad -(16)$$

At $T = 2.7K$:

$$N = 4.0 \times 10^{8}\ \text{photons}\ m^{-3} \quad -(17)$$

Eq. (8) can be used to evaluate the mean square mass for a given frequency and temperature.
