

243(5) : The Cumulative Capacity of the Einstein Solid
 In note 243(4) it was shown that the mean \bar{E} of the Einstein solid is:

$$\langle E \rangle = \frac{1}{k_B} \langle E^2 \rangle \quad - (1)$$

where $\langle E^2 \rangle$ is the mean square energy. The latter is quantized and given by:

$$\langle E^2 \rangle = \frac{\sum_n E_n^2 \exp\left(-\frac{E_n}{k_B T}\right)}{\sum_n \exp\left(-\frac{E_n}{k_B T}\right)} \quad - (2)$$

as in note 243(4). The square of energy is quantized by:

$$E^2 = 0, k_B^2 \omega^2, 2 k_B^2 \omega^2, \dots, n k_B^2 \omega^2 \quad - (3)$$

so

$$\langle E^2 \rangle = \frac{\sum_n n k_B^2 \omega^2 e^{-x}}{\sum_n e^{-x}} \quad - (4)$$

where

$$x := \exp\left(-\frac{k_B \omega}{k_B T}\right) \quad - (5)$$

Note that:

$$\sum_n x^n = (1-x)^{-1} = 1 + x + x^2 + \dots \quad - (6)$$

and:

$$\sum_n n x^n = x \frac{d}{dx} \sum_n x^n \quad - (7)$$

$$\begin{aligned} \text{So } \langle E^2 \rangle &= f^2 \omega^2 x \frac{d}{dx} (1-x)^{-1} / (1-x)^{-1} \\ &= f^2 \omega^2 \frac{x}{1-x} \quad - (8) \end{aligned}$$

Therefore

$$\langle R \rangle = \frac{\omega^2}{c^2} \frac{x}{1-x} \quad - (9)$$

is the mean curvature of the Einstein solid.

The capacity of mean curvature is :

$$\begin{aligned} C_V(R) &= 3N \frac{d \langle R \rangle}{dT} \quad - (10) \\ &= \frac{3N}{f^2 c^2} \frac{d \langle E \rangle}{dT} \end{aligned}$$

$$= 3N \frac{\omega^2}{c^2} \frac{d}{dT} \left(\frac{x}{1-x} \right),$$

$$\begin{aligned} C_V(R) &= \frac{3N f \omega^3}{k T^2 c^2} \frac{\exp \left(\frac{f \omega}{k T} \right)}{\left(1 - \exp \left(\frac{f \omega}{k T} \right) \right)^2} \\ &\quad - (11) \end{aligned}$$

The density of states is:

$$dN = \frac{8\pi f^2}{c^3} df \quad - (12)$$

in inverse metres cubed, where

$$\omega = 2\pi f \quad - (13)$$

so

$$dN = \frac{\omega^2 d\omega}{\pi c^3} \quad - (14)$$

The Planck distribution of quantized space of energy is:

$$dU^2 = \langle E^2 \rangle dN \quad - (15)$$

$$= h^2 \omega^2 \frac{x}{1-x} \cdot \frac{\omega^2}{\pi c^3} d\omega$$

$$dU^2 = \frac{h^2 \omega^4}{\pi c^3} \left(\frac{x}{1-x} \right) d\omega \quad - (16)$$

The total space of energy of black body radiation is:

$$U^2 = \int_0^\infty \frac{h^2 \omega^4}{\pi c^3} \left(\frac{x}{1-x} \right) d\omega \quad - (17)$$