## 243(5): The Curvature Capacity of the Einstein Solid of the mean ECE

In note 243(3) it was shown that the mean curvature of the Einstein solid is:

$$\langle R \rangle = \frac{1}{\hbar^2 c^2} \langle E^2 \rangle \quad -(1)$$

where $\langle E^2 \rangle$ is the mean square energy. The latter is quantized and given by:

$$\langle E^2 \rangle = \frac{\sum_n E_n^2 \exp\left(-\frac{E_n}{kT}\right)}{\sum_n \exp\left(-\frac{E_n}{kT}\right)} \quad -(2)$$

As in note 243(4) the square of energy is quantized by:

$$E^2 = 0,\ \hbar^2\omega^2,\ 2\hbar^2\omega^2,\ \ldots,\ n\hbar^2\omega^2 \quad -(3)$$

So

$$\langle E^2 \rangle = \frac{\sum_n n\hbar^2\omega^2 e^{-x}}{\sum_n e^{-x}} \quad -(4)$$

where

$$x := \exp\left(-\frac{\hbar\omega}{kT}\right) \quad -(5)$$

Note that:

$$\sum_n x^n = (1-x)^{-1} = 1 + x + x^2 + \ldots \quad -(6)$$

and:

$$\sum_n n x^n = x \frac{d}{dx} \sum_n x^n \quad -(7)$$

So

$$\langle E^2 \rangle = \hbar^2 \omega^2 x \frac{d}{dx} (1-x)^{-1} / (1-x)^{-1}$$

$$= \hbar^2 \omega^2 \frac{x}{1-x} \quad -(8)$$

Therefore

$$\boxed{\langle R \rangle = \frac{\hbar \omega^2}{c^2} \frac{x}{1-x}} \quad -(9)$$

is the mean curvature of the Einstein solid.

The capacity of mean curvature is:

$$C_v(R) = 3N \frac{d \langle R \rangle}{dT} \quad -(10)$$

$$= \frac{3N}{\hbar^2 c^2} \frac{d \langle E^2 \rangle}{dT}.$$

$$= 3N \frac{\hbar \omega^2}{c^2} \frac{d}{dT} \left( \frac{x}{1-x} \right),$$

i.e

$$\boxed{C_v(R) = \frac{3N \hbar^2 \omega^3}{k T^2 c^2} \frac{\exp\left(\frac{\hbar \omega}{kT}\right)}{\left(1 - \exp\left(\frac{\hbar\omega}{kT}\right)\right)^2}} \quad -(11)$$

)

The density of states is:

$$dN = \frac{8\pi f^2}{c^3} df \quad -(12)$$

in inverse metres cubed, where

$$\omega = 2\pi f \quad -(13)$$

so

$$dN = \frac{\omega^2 \, d\omega}{\pi^2 c^3} \quad -(14)$$

The Planck distribution of the quantized square of energy is:

$$du^2 = \langle E^2 \rangle dN \quad -(15)$$

$$= \hbar^2 \omega^2 \frac{x}{1-x} \cdot \frac{\omega^2}{\pi^2 c^3} d\omega$$

i.e

$$\boxed{du^2 = \frac{\hbar^2 \omega^4}{\pi^2 c^3} \left( \frac{x}{1-x} \right) d\omega} \quad -(16)$$

The total square of energy of black body radiation is:

$$\boxed{u^2 = \int_0^\infty \frac{\hbar^2 \omega^4}{\pi^2 c^3} \left( \frac{x}{1-x} \right) d\omega} \quad -(17)$$