242(6): Comparison of Expressions for Time

For any curve:

$$dA = \frac{1}{2} r^2 d\theta \quad -(1)$$

where A is area. Assume that in a time T an area A is swept out. Then in a time t an area At/T is swept out. Therefore

$$\frac{A}{T} t = \int^{A} dA = \frac{1}{2} \int_0^{\theta} r^2 d\theta \quad -(2)$$

therefore

$$t = \frac{T}{2A} \int_0^{\theta} r^2 d\theta \quad -(3)$$

In the Newtonian theory the curve is an ellipse of area:

$$A = \pi ab = \pi \alpha^2 (1-\epsilon^2)^{-3/2} \quad -(4)$$

and

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(5)$$

$$\frac{dr}{d\theta} = \frac{\epsilon r^2 \sin\theta}{\alpha} \quad -(6)$$

From computer algebra:

$$t = \frac{1}{\sqrt{2}} \int \left(- \int r \Omega^2 dr - x \right)^{-1/2} dr + y \quad -(7)$$

where x and y are constants of integration.
Comparing eqs (3) and (7) gives:

2)

$$\frac{1}{\sqrt{2}}\int\left(-\int r\Omega^2\,dr - x\right)^{-1/2} dr + y = \frac{T}{2A}\int_0^\theta r^2\,d\theta \quad -(8)$$

for any curve, and any force law.

Assume for simplicity that:

$$y = 0. \quad -(9)$$

From eq. (6):

$$dr = \frac{\epsilon r^2}{\alpha}\sin\theta\,d\theta \quad -(10)$$

so

$$\frac{1}{\sqrt{2}}\int_0^\theta\left(-\int r\Omega^2\,dr - x\right)^{-1/2} dr = \frac{T}{2A}\int_0^\theta r^2\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^\theta\left(-\int r\Omega^2\,dr - x\right)^{-1/2}\frac{\epsilon r^2}{\alpha}\sin\theta\,d\theta \quad -(11)$$

Therefore:

$$\left(-\int r\Omega^2\,dr - x\right)^{-1/2} = \frac{\sqrt{2}}{2}\,\frac{T}{A}\,\frac{\alpha}{\epsilon\sin\theta}$$

$$= \frac{1}{\sqrt{2}}\,\frac{T}{ab\pi}\,r^2\frac{d\theta}{dr} \quad -(12)$$

In general, let:

3)

$$f(r) = \left(- \int r \Omega^2 dr - x \right)^{-1/2} \quad - (13)$$

and
$$t = \frac{1}{\sqrt{2}} \int f(r) \, dr = \frac{1}{2} \frac{T}{A} \int r^2 d\theta \quad - (14)$$

In general, for any curve and force law:

$$\frac{d\theta}{dr} = f_1(r) \quad - (15)$$

so
$$d\theta = f_1(r) \, dr \quad - (16)$$

and
$$t = \frac{1}{\sqrt{2}} \int f(r) \, dr = \frac{1}{2} \frac{T}{A} \int r^2 f_1(r) \, dr \quad - (17)$$

so in general: $\quad - (18)$

$$\boxed{f(r) = \left(- \int r \Omega^2 dr - x \right)^{-1/2} = \frac{1}{\sqrt{2}} \frac{T}{A} r^2 f_1(r)}$$

where
$$f_1(r) = \frac{d\theta}{dr} \quad - (19)$$

Therefore:
$$\frac{d\theta}{dr} = \sqrt{2} \frac{A}{T} f(r) \quad - (20)$$

and
$$\theta = \sqrt{2} \frac{A}{T} \int \frac{f(r)}{r^2} dr \quad - (21)$$

4) General Result

The transcribed angle θ for any orbit and any force law is:

$$\theta = \sqrt{2}\,\frac{A}{T}\int \frac{f(r)}{r^2}\,dr \quad -(22)$$

where

$$f(r) = \left(-\int r\Omega^2 dr - x\right)^{-1/2} \quad -(23)$$

and

$$\Omega^2 = -\frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr} \quad -(24)$$

where A is the area swept out in the time T needed for one orbit. Here:

$$\frac{d^2 r}{dt^2} + \Omega^2 r = 0 \quad -(25)$$

Units Check

The units of Ω^2 are s^{-2}, so the units of $f(r)$ are sm^{-1}. Therefore the units in eq. (22) are correct.

Conclusion

The precession of the perihelion can be calculated for any force law.