# 242(1): The Apsidal Angle for Any Planar Orbit.

Consider the orbit of a mass $m$ around a mass $M$ in a plane. The two equations of motion are:

$$m(\ddot{r} - r\dot{\theta}^2) = F(r) \quad -(1)$$

and

$$L = mr^2\dot{\theta} \quad -(2)$$

From eq. (2) in eq. (1):

$$m\left(\ddot{r} - \frac{rL^2}{m^2r^4}\right) = F(r) \quad -(3)$$

so

$$\ddot{r} - \frac{L^2}{m^2r^3} = \frac{F(r)}{m} \quad -(4)$$

i.e.

$$\boxed{\ddot{r} - \omega^2 r = \frac{F(r)}{m}} \quad -(5)$$

i.e.

$$\frac{d^2r}{dt^2} + \left(-\omega^2 - \frac{F(r)}{mr}\right)r = 0 \quad -(6)$$

This is a harmonic oscillator with period:

$$T = 2\pi\left(-\omega^2 - \frac{F(r)}{mr}\right)^{-1/2} \quad -(7)$$

The apsidal angle is:

2)

$$\phi = \frac{1}{2} T \frac{d\theta}{dt} = \frac{1}{2} \omega T \quad -(8)$$

where

$$\omega = \frac{L_0}{mr^2} \quad -(9)$$

so

$$\boxed{\phi = \frac{\pi L_0}{mr^2}\left(-\frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr}\right)^{-1/2}} \quad -(10)$$

and

$$\theta = 2\phi \quad -(11)$$