242(3): Equation for θ for any Planar Orbit

From a consideration of Lagrangian dynamics:

$$\frac{d^2 r}{dt^2} + \Omega^2(r)\, r = 0 \quad - (1)$$

where

$$\Omega^2 = -\frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr} \quad - (2)$$

in which the conserved angular momentum is:

$$L_0 = m r^2 \frac{d\theta}{dt} \quad - (3)$$

Computer algebra shows that eq. (1) has the general solution:

$$t = \frac{1}{\sqrt{2}} \int \left(- \int r\,\Omega^2(r)\, dr - k1 \right)^{-1/2} dr + k_2$$

$$- (4)$$

Assuming that the constants of integration k1 and k2 are zero:

$$\boxed{t = \frac{1}{\sqrt{2}} \int \left(- \int r\,\Omega^2(r)\, dr \right)^{-1/2} dr \quad - (5)}$$

For any circle:

$$dt = \frac{2m}{L_0}\, dA \quad - (6)$$

in Lagrangian dynamics, where A is its area.

2) The angle θ transcribed in time T is:

$$\theta = \omega T = \frac{L_0}{mr^2} T = \frac{2A}{r^2} \quad -(7)$$

because in Lagrangian dynamics:

$$T = \int dt = \frac{2m}{L_0} A \quad -(8)$$

(Marion and Thornton, 3rd ed., page 259).

For a circle:

$$\theta = 2\pi, \quad A = \pi r^2. \quad -(9)$$

For an ellipse:

$$\theta = 2\pi \frac{ab}{r^2}, \quad A = \pi ab. \quad -(10)$$

Consider eq. (5) for a time interval T, then using eq. (8):

$$T = \frac{2m}{L_0} A = \frac{1}{\sqrt{2}} \int \left(- \int r \Omega^2(r) dr \right)^{-1/2} dr \quad -(11)$$

so

$$A = \frac{L_0}{2\sqrt{2} m} \int \left(- \int r \Omega^2(r) dr \right)^{-1/2} dr \quad -(12)$$

and

$$\theta = \frac{2A}{r^2} = \frac{L_0}{r^2 \sqrt{2} m} \int \left(- \int r \Omega^2(r) dr \right)^{-1/2} dr \quad -(13)$$

The angle θ transcribed by the distance R along the planar orbit is:

3)

$$\boxed{\theta = \frac{L_0}{\sqrt{2}\, m R^2} \int_0^R \left(\int_0^R - r\Omega^2(r) \right)^{-1/2} dr} \quad -(14)$$

From eqs. (2) and (14), θ may be worked out for any force law $F(r)$.

1) The Newtonian force law is:

$$F(r) = -\frac{mMG}{r^2} \quad -(15)$$

The Einstein force law is:

2) $$F(r) = -\frac{mMG}{r^2} - \frac{3L_0^2 MG}{m^2 c^2 r^4} \quad -(16)$$

3) The force law of the true precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(17)$$

is

$$F(r) = -\frac{mMGx^2}{r^2} - \alpha(1 - x^2)\frac{mMG}{r^3} \quad -(18)$$

4) The approximate Minkowski force law is:

4)

$$F = -\gamma^4 \frac{mMG}{r^2} - \gamma^2 \frac{\alpha mMG}{r^3}\left(1-\gamma^2\right) \quad -(19)$$

$$\gamma^2 = \left(1 - \frac{MG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)\right)^{-1} \quad -(20)$$

For a given R, the angle θ can be worked out from eq. (14) using these four different force laws. The double integral in eq. (14) is probably not analytical, but can be worked out to machine precision with an integration routine. The precession of the perihelion can be worked out exactly and this would check the claims of the Einstein ~~theory~~.

The numerical method can be checked with a circular orbit, where:

$$\alpha = r, \quad L_0^2 = \alpha m^2 MG \quad -(21)$$
$$= r m^2 MG$$

$$\Omega_0^2 = -\frac{2MG}{r^3} \quad -(22)$$

So

$$\theta = \frac{L_0}{\sqrt{2}\, m r^2} \int \left(-\int r\,\Omega^2(r)\,dr\right)^{-1/2} dr \quad -(23)$$