241(5): Expression for θ in Terms of t.

From note 241(4):

$$t = \frac{m\alpha}{\epsilon L_0} \int \frac{1}{x} \left(1 - \left(\frac{\alpha}{r} - 1 \right)^2 \right)^{-1/2} dr, \quad -(1)$$

where

$$x^2 = \frac{r\gamma^4 + \alpha\gamma^2(1-\gamma^2) - \alpha}{r - \alpha}, \quad -(2)$$

$$\gamma^2 = \left(1 - \frac{\alpha M G}{r^2 c^2} \left(1 + 2\left(\frac{\alpha}{r}\right) - \left(\frac{\alpha}{r}\right)^2 \right) \right)^{-1}$$

$-(3)$.

Computer algebra shows that:

$$r = A^{1/3} + \frac{\alpha^3}{8} A^{-1/3} \quad -(4)$$

where

$$A = \left(\frac{3\epsilon t x L_0}{4\alpha m^2} \left(9\epsilon^2 t^2 x^2 L_0^2 + \alpha^4 m^2 \right)^{1/2} + \frac{18\epsilon^2 t^2 x^2 L_0^2 + \alpha^4 m^2}{8\alpha m^2} \right)^{1/3} \quad -(5)$$

So r is known as a function of t.

The equation relating r to θ is:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(6)$$

so:

2)

$$\theta = \frac{1}{x}\cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right)\right) \quad -(7)$$

So from eqs. (4) and (7) θ can be calculated as a function of t. As in Marion and Thornton 3rd edition, pp. 261 ff. calculations in astronomy are based on $\theta(t)$, so the trajectory θ can be found at any time t.

As in note 238(12):

$$\underline{r}(\theta) = r(\theta)\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}\right) \quad -(8)$$

where:

$$X = \frac{\alpha\cos\theta}{1+\epsilon\cos(x\theta)}, \quad Y = \frac{\alpha\sin\theta}{1+\epsilon\cos(x\theta)} \quad -(9)$$

Since θ is known as a function of t, X and Y are also known as a function of t, and so $(X(t), Y(t))$ can be animated.