241(3): Precession due to the Minkowski Equation.

From note 239(6): $\quad -(1)$

$$\underline{F} = m\left(\gamma^4 \frac{d^2 r}{dt^2} - \frac{\gamma^2 L_0^2}{m^2 r^3}\right)\underline{e}_r + \frac{m\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2}\,\omega r\,\underline{e}_\theta$$

As in note 238(3):

$$\frac{dr}{dt} = -\frac{L_0}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right), \quad \frac{d^2 r}{dt^2} = -\left(\frac{L_0}{mr}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(2)$$

and

$$L_0 = m r^2 \omega. \quad -(3)$$

So:

$$\underline{F} = -\frac{\gamma^2 L_0^2}{m r^2}\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r + \left(\frac{\gamma^4 L_0^4}{m^3 c^2 r^3}\right)\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta . \quad -(3)$$

Here:

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(4)$$

where

$$v^2 = \left(\frac{L_0}{m}\right)^2\left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right). \quad -(5)$$

Assume that:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad -(6)$$

so:

2) $$v^2 = \left(\frac{L_0}{m\alpha}\right)^2 \left(1+\epsilon^2 + 2\epsilon\cos\theta\right) \quad - (7)$$

where $$\epsilon\cos\theta = \frac{\alpha}{r} - 1, \quad - (8)$$

and in the Newtonian limit:
$$L_0^2 = \alpha m^2 \underline{M} G. \quad - (9)$$

So $$v^2 = \frac{\underline{M}G}{\alpha}\left(1+\epsilon^2 + 2\left(\frac{\alpha}{r} - 1\right)\right) \quad - (10)$$
$$= \underline{M}G\left(\frac{2}{r} - \frac{1}{a}\right) \quad - (11)$$

where $$a = \frac{\alpha}{\epsilon^2 - 1} \quad - (12)$$

Therefore:
$$\underline{F} = A\underline{e}_r + B\underline{e}_\theta \quad - (13)$$

where $$A = -\frac{\gamma^4 L_0^2}{m r^2 \alpha} - \frac{\gamma^2 L_0^2 (1-\gamma^2)}{m r^3} \quad - (14)$$
$$= -\frac{\gamma^2 L_0^2}{m r^2}\left(\gamma^2 + \frac{1}{r}\left(1-\gamma^2\right)\right)$$

using eq. (9):

3)
$$A = -\gamma^4 \frac{mMG}{r^2} - \gamma^2 \frac{\alpha m M G}{r^3}(1-\gamma^2) \quad - (15)$$

and
$$B = \frac{\gamma^4 L_0^4 \sin\theta \cos\theta}{m^3 c^2 r^3 \alpha^2} \quad - (16)$$

For a nearly circular orbit:
$$a \sim r \quad - (17)$$

so
$$v^2 \sim \frac{MG}{r} \quad - (18)$$

In this case:
$$\gamma^2 = \left(1 - \frac{MG}{c^2 r}\right)^{-1}, \quad - (19)$$
$$\gamma^4 = \left(1 - \frac{MG}{c^2 r}\right)^{-2} \quad - (20)$$

The apsidal angle in the limit of nearly circular orbits is:
$$\phi = \pi\left(3 + \frac{r}{F}\frac{dF}{dr}\right)^{-1/2} \quad - (21)$$

and
$$\theta = 2\phi. \quad - (22)$$

In the limit of nearly circular orbits:

4)

$$\alpha \to r \quad -(23)$$

So

$$A \to -\gamma^4 \frac{mMG}{r^2}, \quad -(24)$$

$$B \to 0 \quad -(25)$$

For

$$MG << c^2 r \quad -(26)$$

$$\gamma^4 \to 1 + \frac{2MG}{c^2 r} = 1 + \frac{r_0}{r} \quad -(27)$$

where

$$r_0 = \frac{2MG}{c^2} \quad -(28)$$

Therefore:

$$\boxed{F \to -\frac{mMG}{r^2}\left(1 + \frac{r_0}{r}\right)} \quad -(29)$$

and

$$\boxed{\Delta\theta = \pi \frac{r_0}{r}} \quad -(30)$$

Eq. (29) is the force law of the correct precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(31)$$

5)

i.e.

$$F = -\frac{mMGx^2}{r^2} - \alpha(1-x^2)\frac{mMG}{r^3} \quad -(32)$$

Eqs. (29) and (32) are the same if:

$$x \to 1 \quad -(33)$$

and:

$$\alpha(1-x^2)mMG = 2m\left(\frac{MG}{c}\right)^2 \quad -(34)$$

i.e.

$$x^2 = 1 - \frac{r_0}{\alpha} \quad -(35)$$

$$\to 1 - \frac{r_0}{r}$$

and

$$x \sim 1 - \frac{r_0}{2r} \quad -(36)$$

so

$$1 - x \sim \frac{r_0}{2r}. \quad -(37)$$

The precession of the perihelia is:

$$2\pi(1-x) = \pi\frac{r_0}{r} \quad -(38)$$

which is eq. (30), Q.E.D.