241(6): Calculation of the Precession Constant in Special Relativity and Animation

As in note 239(7) the precession constant x can be calculated by extending the definition of angular momentum. For two dimensional orbits the non-relativistic angular momentum is:

$$L_0 = mr^2 \frac{d\theta}{dt} \quad -(1)$$

and it follows that:

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2}. \quad -(2)$$

The relativistic angular momentum is:

$$L = mr^2 \frac{d\theta}{d\tau} \quad -(3)$$

$$= \gamma L_0$$

where τ is the proper time and γ the Lorentz factor:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(4)$$

It follows that:

$$dt = \frac{mr^2}{L_0} d\theta \quad -(5)$$

$$d\tau = \frac{mr^2}{\gamma L_0} d\theta. \quad -(6)$$

Now rewrite eq. (6) as:

$$d\tau = \frac{mr^2}{L_0}\left(\frac{d\theta}{\gamma}\right) \quad -(7)$$

Therefore the change:

$$dt \rightarrow d\tau \quad -(8)$$

2) is produced by:

$$d\theta \rightarrow \frac{d\theta}{\gamma} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} d\theta \quad -(9)$$

For a revolution of 2π radians:

$$\int_0^{2\pi} d\theta \longrightarrow \int_0^{2\pi} \left(1 - \frac{v^2}{c^2}\right)^{1/2} d\theta \quad -(10)$$

i.e.

$$2\pi \rightarrow \int_0^{2\pi} \left(1 - \frac{v^2}{c^2}\right)^{1/2} d\theta \quad -(11)$$

If the orbit is initially an ellipse, then the velocity is in note 238(4):
case expressed in terms of θ as

$$v^2 = \left(\frac{L_0}{m\alpha}\right)^2 \left(1 + \epsilon^2 + 2\epsilon \cos\theta\right) \quad -(12)$$

where

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(13)$$

in Newtonian theory (the non-relativistic limit).
The half right latitude is:

$$\alpha = (1+\epsilon) r_{min} = (1-\epsilon) r_{max} = (1-\epsilon^2) a = (1-\epsilon^2)^{1/2} b \quad -(14)$$

where a is the semimajor axis, b the semi-minor axis and r_{max} and r_{min} are the maximum and minimum

3) distances of an orbiting mass m from a mass M at one focus of the ellipse.

The relativistic effect of eq. (8) therefore produces a change of angle:

$$\Delta\theta = 2\pi(1-x) \quad -(15)$$

so:

$$x = \frac{1}{2\pi}\int_0^{2\pi}\left(1-\left(\frac{L_0}{mc\alpha}\right)^2\left(1+\epsilon^2+2\epsilon\cos\theta\right)\right)^{1/2}d\theta \quad -(16)$$

and the ellipse becomes the precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(17)$$

Therefore special relativity is enough to produce a precessing ellipse.

In the Newtonian theory:

$$L_0^2 = m^2 M G\alpha \quad -(18)$$

so:

$$x = \frac{1}{2\pi}\int_0^{2\pi}\left(1-\frac{MG}{\alpha c^2}\left(1+\epsilon^2+2\epsilon\cos\theta\right)\right)^{1/2}d\theta \quad -(19)$$

The old "Schwarzschild" radius is defined as:

$$r_0 = \frac{2MG}{c^2} \quad -(20)$$

4) So:

$$x = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \frac{r_0}{2\alpha} \left(1 + \epsilon^2 + 2\epsilon \cos\theta \right) \right)^{1/2} d\theta \quad -(21)$$

Animation without Approximation

The method used in note 238(12) can be extended to animate the precessing ellipse with x given by eq. (21).

The equation for time t is given by:

$$t = \frac{m\alpha^2}{L_0} \int \frac{d\theta}{(1 + \epsilon \cos(x\theta))^2} \quad -(22)$$

where x is given by the definite integral (21).

The Cartesian coordinates are given by:

$$X = \frac{\alpha \cos\theta}{1 + \epsilon \cos(x\theta)}, \quad Y = \frac{\alpha \sin\theta}{1 + \epsilon \cos(x\theta)} \quad -(23)$$

with x given by eq. (21).

Plot (X, Y) as a function of θ and animate.

For each θ calculate t from eq. (22), plot (X, Y) as a function of t and animate.

5) For example:

$$t_1 = \frac{m\alpha^2}{L_0} \int_0^{\theta_1} \frac{d\theta}{(1+\epsilon\cos(x\theta))^2}, \quad -(24)$$

$$x = \frac{1}{2\pi}\int_0^{2\pi}\left(1 - \frac{r_0}{2\alpha}\left(1+\epsilon^2+2\epsilon\cos\theta\right)\right)^{1/2} d\theta,$$

$$X_1 = \frac{\alpha\cos\theta_1}{1+\epsilon\cos(x_1\theta_1)}, \quad Y_1 = \frac{\alpha\sin\theta_1}{1+\epsilon\cos(x_1\theta_1)}$$

$$-(25)$$

$$x_1 = \frac{1}{2\pi}\int_0^{2\pi}\left(1 - \frac{r_0}{2\alpha}\left(1+\epsilon^2+2\epsilon\cos\theta_1\right)\right)^{1/2} d\theta_1.$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(1 - \frac{r_0}{2\alpha}\left(1+\epsilon^2+2\epsilon\cos\theta\right)\right)^{1/2} d\theta$$

and animate using: (X_1, Y_1) at t_1; (X_2, Y_2) at t_2; ...; (X_n, Y_n) at t_n.

As in note 239(7), if:

$$\frac{r_0}{2\alpha} \ll 1 \quad -(26)$$

then:

$$x \sim 1 - \frac{r_0(1+\epsilon^2)}{4\alpha} \quad -(27)$$

6) and the algorithm is simplified because x is a constant.

For the earth sun system:

$$r_0 = 2.95 \times 10^3 \; m$$

$$\epsilon = 0.0167$$

$$r_{max} = 1.521 \times 10^{11} \; m$$

$$r_{min} = 1.471 \times 10^{11} \; m$$

$$d = 1.496 \times 10^{11} \; m$$

and

$$\Delta\theta \sim 0.64'' \text{ per century.}$$

The precession given by special relativity is this; note is:

$$\Delta\theta = 2\pi\left(1 - \int_0^{2\pi}\left(1 - \frac{v^2}{c^2}\right)^{1/2} d\theta\right) \quad -(28)$$

and is very similar to the Thomas precession calculated in UFT110, eq. (17.13):

$$\Delta\theta = 2\pi\left(1 - \left(1 - \frac{v^2}{c^2}\right)^{1/2}\right) \quad -(29)$$