240(2): Calculation of Perihelion Precession for any Force Law.

Consider the lagrangian for orbital motion in a plane:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \quad -(1)$$

where the reduced mass is:

$$\mu = \frac{mM}{m+M} \quad -(2)$$

If

$$m << M \quad -(3)$$

then

$$\mu \sim m. \quad -(4)$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(5)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(6)$$

The total angular momentum is conserved and is:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}. \quad -(7)$$

Fm eq. (6):

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(r)}{m} := f(r) \quad -(8)$$

Fm eq. (7):

$$h = \frac{L}{m} = r^2\dot{\theta} = \text{constant} \quad -(9)$$

) Consider small deviations:

$$x = r - r_{av} \quad -(10)$$

from a nearly circular orbit, such as an orbit of a planet in the solar system. Then from eqs. (8) and (9):

$$\ddot{x} - \frac{h^2}{(r_{av} + x)^3} = f(r_{av} + x) \quad -(11)$$

For small x, a Maclaurin expansion gives:

$$\ddot{x} - \frac{h^2}{r_{av}^3}\left(1 - \frac{3x}{r_{av}}\right) \sim f(r_{av}) + \frac{\partial f(r_{av})}{\partial r_{av}} x \quad -(12)$$

For a nearly circular orbit:

$$\ddot{r} \sim 0 \quad -(13)$$

so eq. (8) becomes:

$$-\frac{h^2}{r_{av}^3} \sim f(r_{av}) \quad -(14)$$

From eqs. (12) and (14):

$$\ddot{x} + \left(-\frac{3f(r_{av})}{r_{av}} - \frac{\partial f(r_{av})}{\partial r_{av}}\right) x = 0 \quad -(15)$$

This is a harmonic oscillator equation with period of oscillation:

3)

$$T = 2\pi \left(-\frac{3f}{r_{av}} - \frac{\partial f}{\partial r} \right)^{-1/2} \quad -(16)$$

In the approximation (14):

$$\dot{\theta} \sim \frac{h}{r_{av}^2} = \left(-\frac{f(r_{av})}{r_{av}} \right)^{1/2} \quad -(17)$$

The angle by which θ increases between a maximum and a minimum of r is the apsidal angle. The time needed for ψ is $T/2$. The apsidal angle for elliptical orbits is π.

For laws of attraction such as:

$$f(r) = -cr^n \quad -(18)$$

the apsidal angle has special properties. In general:

$$\phi = \frac{1}{2} T \dot{\theta} \quad -(19)$$

so:

$$\boxed{\phi = \pi \left(3 + \frac{r_{av}}{f} \frac{\partial f}{\partial r_{av}} \right)^{-1/2}} \quad -(20)$$

In order for the orbit to be closed the apsidal angle has to be a rational function of ϕ. From

4) eqns. (18) and (20):

$$\phi = \frac{\pi}{(3+n)^{1/2}}, \quad -(21)$$

and for the inverse square law:

$$n = -2 \quad -(22)$$

then

$$\phi = \pi \quad -(23)$$

QED, the orbit is elliptical.

To work this out in detail, for the inverse square law:

$$f = \frac{c}{r_{av}^2} \quad -(24)$$

and

$$\frac{\partial f}{\partial r_{av}} = -\frac{2c}{r_{av}^3} \quad -(25)$$

so

$$\frac{r_{av}}{f}\frac{\partial f}{\partial r_{av}} = -2 \quad -(26)$$

and

$$\phi = \pi \quad -(27)$$

QED. The Einstein theory uses:

$$f(r) = -\frac{k}{r^2} - \frac{\epsilon}{r^4} \quad -(28)$$

5)

So:

$$\phi = \pi\left(3 + r_{av}\left(\frac{2kr_{av}^{-3} + 4\epsilon r_{av}^{-5}}{-kr_{av}^{-2} - \epsilon r_{av}^{-4}}\right)\right)^{-1/2}$$

$$= \pi\left(3 - 2\left(\frac{1 + 2\epsilon/(kr_{av}^2)}{1 + \epsilon/(kr_{av}^2)}\right)\right)^{-1/2} \quad -(29)$$

If:

$$\epsilon \ll kr_{av}^2 \quad -(30)$$

then:

$$\phi \sim \pi\left(3 - 2\left(1 + \frac{\epsilon}{kr_{av}^2}\right)\right)^{-1/2}$$

$$= \pi\left(1 - \frac{2\epsilon}{kr_{av}^2}\right)^{-1/2}$$

$$\sim \pi\left(1 + \frac{\epsilon}{kr_{av}^2}\right) \quad -(31)$$

The apsidal angle advances by:

$$\boxed{\Delta\phi = \frac{\pi\epsilon}{kr_{av}^2}} \quad -(32)$$

in the approximation (30).

6) In the Einstein theory:

$$k = mG, \quad \epsilon = \frac{3L_0^2 mG}{m^2 c^2} \quad -(33)$$

and

$$\Delta\phi \sim 3\pi \left(\frac{L_0}{mc\,r_{av}}\right)^2 \quad -(34)$$

$$= 3\pi \left(\frac{\alpha M G}{c^2 r_{av}^2}\right)$$

In one complete revolution of 2π the perihelion advance is $2\Delta\phi$:

$$\Delta\theta = 2\Delta\phi = \frac{6\pi MG}{c^2} \frac{\alpha}{r_{av}^2} \quad -(35)$$

Now use:

$$r_{min} = \frac{\alpha}{1+\epsilon} \quad -(36)$$

$$r_{max} = \frac{\alpha}{1-\epsilon} \quad -(37)$$

and so for an approximately circular orbit:

$$r_{av} \sim r_{min} = r_{max} = \alpha \quad -(38)$$

$$\epsilon \sim 0 \quad -(39)$$

so:

$$\Delta\theta \sim \frac{6\pi MG}{\alpha c^2} \quad -(40)$$

7) For an elliptical orbit:

$$d = (1-\epsilon^2)\,a \quad -(41)$$

and so

$$d \sim a \quad -(42)$$

This gives:

$$\Delta\theta = \frac{6\pi MG}{ac^2} \quad -(43)$$

The result given in Marion and Thornton is

$$\Delta\theta = \frac{6\pi MG}{ac^2(1-\epsilon^2)} \quad -(44)$$

For the earth:

$$\epsilon = 0.0167 \quad -(45)$$

$$\epsilon^2 = 2.79\times10^{-4} \quad -(46)$$

so eq. (43) is adequate.

However eq. (43) is only an approximation to eq. (29), which is itself based on an approximation of circular orbits.

In previous work:

$$\theta = 2\pi(1+x) \quad -(47)$$

$$\Delta\theta = 2\pi x$$

So

$$\boxed{x = \frac{\alpha \epsilon}{k r_{av}^2}} \quad -(48)$$

But it is known from previous work that the Einstein force law (28) does not produce the true precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(49)$$

The force law needed for eq. (49) is the sum of terms inverse square and cube in r.

Using the new method of this note, x can be found analytically for the true precessing ellipse (49). This will be the subject of the next note. Furthermore, any force law:

$$f(r) = -\left(\frac{A}{r^2} + \frac{B}{r^3} + \dots + \frac{Y}{r^n}\right) \quad -(50)$$

will produce precession.
