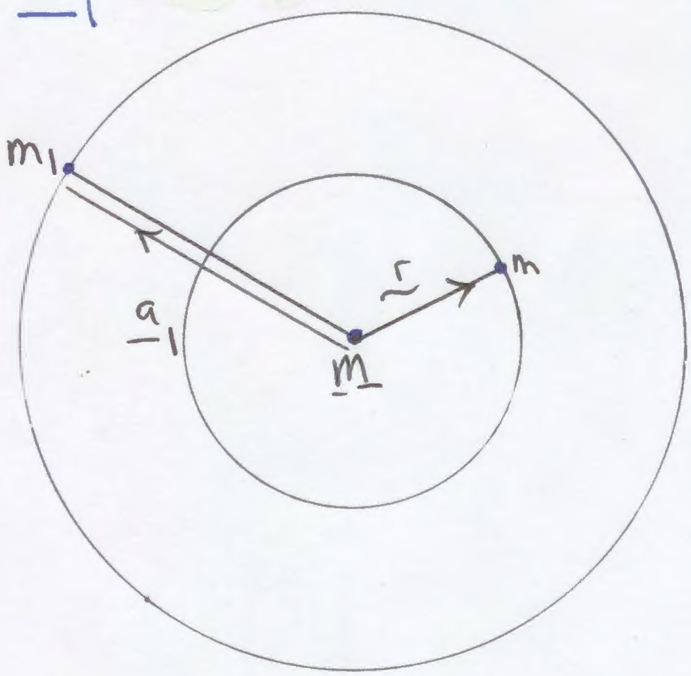
240(b): Calculation of the Perihelion Precession of Mercury due to Venus and other Planets

Let the mass of the sun be M, the mass of Mercury be m and the mass of Venus m_1. Let the distance of Mercury from the sun be $|\underset{\sim}{r}|$, and that of Venus from the sun be $|\underline{a}_1|$.



The gravitational potential is:

$$\phi = -\frac{mMG}{|\underline{r}|} - \frac{mm_1G}{|\underline{a}_1 - \underline{r}|} \quad -(1)$$

where

$$a_1 > r \quad -(2).$$

The force is

$$F = -\frac{\partial\phi}{\partial r} \quad -(3)$$

and the orbital angle of the orbit of Mercury is:

2)

$$\phi \sim \pi \left(3 + \frac{r}{F}\frac{dF}{dr} \right)^{-1/2} \quad -(4)$$

for nearly circular orbits.

Expanding eq. (1) in Legendre polynomials:

$$\phi(r) = -\frac{mMG}{r} - \frac{mm_1G}{a_1}\left(1 + \frac{1}{4}\left(\frac{r}{a_1}\right)^2 + \frac{9}{64}\left(\frac{r}{a_1}\right)^4 + \dots\right) \quad -(5)$$

for:

$$a_1 > r. \quad -(6)$$

Note carefully that this is a Newtonian calculation. It produces precession of the perihelion of Mercury as follows.

The force from eqs. (3) and (5) is:

$$F(r) = -\frac{mMG}{r^2} + \frac{mm_1G}{a_1^2}\left(\frac{1}{2}\left(\frac{r}{a_1}\right) + \frac{9}{16}\left(\frac{r}{a_1}\right)^3\right) \quad -(7)$$

Therefore:

$$\frac{dF(r)}{dr} = \frac{2mMG}{r^3} + \frac{mm_1G}{a_1^3}\left(\frac{1}{2} + \frac{27}{16}\left(\frac{r}{a_1}\right)^2\right) \quad -(8)$$

Now denote:

3)

$$x := \frac{m m_1 G}{a_1^3}\left(\frac{1}{2} + \frac{27}{16}\left(\frac{r}{a_1}\right)^2\right) \quad -(9)$$

$$y := \frac{m m_1 G}{a_1^2}\left(\frac{1}{2}\left(\frac{r}{a_1}\right) + \frac{9}{16}\left(\frac{r}{a_1}\right)^3\right) \quad -(10)$$

then

$$\phi = \pi\left(\frac{3 + \frac{2mMG}{r^2} + xr}{-\frac{mMG}{r^2} + y}\right)^{-1/2} \quad -(11)$$

$$= \pi\left(\frac{-\frac{mMG}{r^2} + 3y + xr}{-\frac{mMG}{r^2} + y}\right)^{-1/2}$$

$$= \pi\left(\frac{1 - \frac{r^2}{mMG}(3y + xr)}{1 - \frac{r^2}{mMG}y}\right)^{-1/2}$$

i.e

$$\phi = \pi\left[\frac{1 - \frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(2 + \frac{54}{16}\left(\frac{r}{a_1}\right)^2\right)}{1 - \frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(\frac{1}{2} + \frac{9}{16}\left(\frac{r}{a_1}\right)^2\right)}\right]^{-1/2}$$

$$-(12)$$

Now denote:

$$x_1 = \frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(2 + \frac{54}{16}\left(\frac{r}{a_1}\right)^2\right), \quad -(13)$$

$$y_1 = \frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(\frac{1}{2} + \frac{9}{16}\left(\frac{r}{a_1}\right)^2\right). \quad -(14)$$

then

$$\phi = \pi\left(1 - x_1\right)^{-1/2}\left(1 - y_1\right)^{1/2} \quad -(15)$$

The mass of Venus is much less than the mass of the sun:

$$\frac{m_1}{M} \ll 1 \quad -(16)$$

so:

$$\phi \sim \pi\left(1 + \frac{1}{2}x_1\right)\left(1 - \frac{1}{2}y_1\right)$$

$$= \pi\left(1 + \frac{1}{2}\left(x_1 - y_1\right) - \frac{1}{4}x_1 y_1\right) \quad -(17)$$

where

$$x_1 - y_1 = \frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(\frac{3}{2} + \frac{45}{16}\left(\frac{r}{a_1}\right)^2\right) \quad -(18)$$

So

$$\phi = \pi\left(1 + \frac{3}{4}\frac{m_1}{M}\left(\frac{r}{a_1}\right)^3 + \frac{45}{32}\frac{m_1}{M}\left(\frac{r}{a_1}\right)^5 - \frac{1}{4}\left(\frac{m_1}{M}\right)^2\left(\frac{r}{a_1}\right)^6\left(2 + \frac{54}{16}\left(\frac{r}{a_1}\right)^2\right)\left(\frac{1}{2} + \frac{9}{16}\left(\frac{r}{a_1}\right)^2\right)\right) \quad -(19)$$

5) To first order in m_1/M:

$$\boxed{\phi \sim \pi\left(1 + \frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(\frac{3}{4} + \frac{45}{32}\left(\frac{r}{a_1}\right)^2\right)\right)}$$

— (20)

The precession of the perihelion of Mercury is therefore:

$$\Delta\theta \sim 2\pi\frac{m_1}{M}\left(\frac{r}{a_1}\right)^3\left(\frac{3}{4} + \frac{45}{32}\left(\frac{r}{a_1}\right)^2\right)$$

— (21)

due to the influence of Venus.

The result (21) depends on the ratio m_1/M of the mass of Venus to that of the sun, and the ratio r/a_1 of the distance of Mercury and Venus from the sun. Eq. (21) is the same as that derived by Fitzpatrick (Farside site)

The numbers in eq. (21) are as follows:

M = mass of sun = 1.99×10^{30} kgm

m_1 = mass of Venus = 4.868×10^{24} kgm

r = distance of Mercury from the sun = 0.579×10^{11} m

a_1 = " " Venus " " " = 1.082×10^{11} m

Therefore to order $(r/a_1)^3$ eq. (21)

This gives: $\Delta\theta = 1.77 \times 10^{-6}$ radian per revolution

$$ -(22) $$

Now use: $1 \text{ radian} = 2.06265 \times 10^{5}$ arc seconds $-(23)$

so.. $\Delta\theta = 0.365''$ per revolution of 2π

$$ -(24) $$

However in astronomy the relevant quantity is the rate of precession. This is given by FitzPatrick as:

$$ \frac{d\,\delta\theta}{dt} = \frac{75}{T}\,\delta\theta \quad -(25) $$

in arc seconds per century. This has units of inverse time.
The time T is defined by:

$$ T = \frac{2\pi}{\omega} \quad -(26) $$

where ω is defined by Kepler's third law:

$$ \omega^2 = \frac{GM}{R^3} \quad -(27) $$

However in his book "Celestial Mechanics", (CUP 2012) FitzPatrick gives a completely different equation (25):

$$ \frac{d\,\delta\theta}{dt} = \frac{1.296 \times 10^{6}}{T}\,\delta\theta \quad -(28) $$

7) because:

$$2\pi \text{ radians} = 1.296 \times 10^{6} \text{ arc-seconds}. \quad -(29)$$

Assuming eq. (28) is correct then the following results are obtained for eq. (21):

planet	m/M	T	R (au)	$\frac{\Delta\theta}{T=1}$ (arcsec / century)	$\frac{\Delta\theta}{T=0.241}$ (arcsec / century)
Mercury	1.66×10^{-7}	0.241	0.387	—	—
Venus	2.45×10^{-6}	0.615	0.723	36.5	151.5
Earth	3.04×10^{-6}	1.000	1.00	17.1	71.0
Mars	3.23×10^{-7}	1.881	1.52	0.5	2.2
Jupiter	9.55×10^{-4}	11.86	5.20	38.3	158.8
Saturn	2.86×10^{-4}	29.46	9.54	1.9	7.7
Uranus	4.36×10^{-5}	84.01	19.19	0.003	0.01
Neptune	5.18×10^{-5}	164.8	30.07	negligible	negligible

Although eq. (21) is exactly the same as that used by Fitzpatrick the results in the above table are different from those given by Fitzpatrick. The reason for this is not clear. However we are now ready to investigate the true effect of EGR.