239(4): Numerical Integration of the Minkowski Force Equation of Orbits

The Minkowski force equation of planar orbits is:

$$\underline{F} = -\frac{\gamma^2 L_0^2}{mr^2}\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right)\underline{e}_r + \frac{\gamma^4 L_0^2}{m^3 r^3 c^2}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta . \quad -(1)$$

In the limit:

$$v << c \quad -(2)$$

this is approximated by:

$$\gamma^4 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{\gamma^2}{r} = -\frac{mr^2 F}{L_0^2} . \quad -(3)$$

Assume that initially:

$$F \sim -\frac{mMG}{r^2} \quad -(4)$$

then:

$$\gamma^4 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{\gamma^2}{r} = \frac{1}{\alpha} \quad -(5)$$

where

$$\alpha = \frac{L_0^2}{m^2 MG} \quad -(6)$$

Therefore

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{\gamma^2 r} = \frac{1}{\gamma^4 \alpha} , \quad -(7)$$

i.e.

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\left(1-\frac{v^2}{c^2}\right) = \frac{1}{\alpha}\left(1-\frac{v^2}{c^2}\right)^2 \quad -(8)$$

2)

i.e.

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}\left(1-\frac{2v^2}{c^2}+\frac{v^4}{c^4}\right)+\frac{v^2}{c^2 r} \quad -(9)$$

The approximation (4) means that the initial assumption is that the orbit is an ellipse. The rigorous equation for an ellipse is:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha} \quad -(10)$$

So assumption (4) is an initial guess for the force in which

$$\gamma \rightarrow 1, \quad -(11)$$

$$v \rightarrow 0. \quad -(12)$$

However eq. (14) of note 238(4) shows that for the ellipse:

$$v^2=\left(\frac{L_0}{m\alpha}\right)^2\left(1+\epsilon^2+2\epsilon\cos\theta\right) \quad -(13)$$

where

$$\epsilon\cos\theta=\frac{\alpha}{r}-1 \quad -(14)$$

so

$$v^2=2\left(\frac{L_0^2}{m^2\alpha r}\right)+(\epsilon^2-1)\left(\frac{L_0^2}{m^2\alpha^2}\right) \quad -(15)$$

The initial guess for the orbit is:

$$r=\frac{\alpha}{1+\epsilon\cos\theta} \quad -(16)$$

and this guess is corrected by eqns. (9) and (15), to produce a new orbit.

3)

Therefore the differential equation of the orbit is:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}-\frac{v^2}{c^2}\left(\frac{2}{\alpha}-\frac{1}{r}\right)+\frac{1}{\alpha}\left(\frac{v}{c}\right)^4 \quad -(17)$$

where
$$v^2=2\left(\frac{L_0^2}{m^2\alpha r}\right)+\left(\epsilon^2-1\right)\left(\frac{L_0^2}{m^2\alpha^2}\right) \quad -(18)$$

and
$$v \ll c. \quad -(19)$$

In the approximation (19):
$$L_0^2=m^2 MG\alpha \quad -(20)$$

so:
$$v^2=2\frac{MG}{r}+\left(\epsilon^2-1\right)\left(\frac{MG}{\alpha}\right) \quad -(21)$$

In the approximation (19), eq. (17) is approximately:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}-\frac{v^2}{c^2}\left(\frac{2}{\alpha}-\frac{1}{r}\right) \quad -(22)$$

$$=\frac{1}{\alpha}-\left(\frac{2}{\alpha}-\frac{1}{r}\right)\frac{1}{c^2}\left(\frac{2MG}{r}+\left(\epsilon^2-1\right)\frac{MG}{\alpha}\right)$$

4) This equation can be expanded as:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}-\frac{1}{c^2}\left(\frac{2(\epsilon^2-1)mG}{\alpha^2}+\frac{4mG}{\alpha r}-(\epsilon^2-1)\frac{mG}{\alpha r}-\frac{2mG}{r^2}\right) \quad -(23)$$

$$=\frac{1}{\alpha}-\frac{2(\epsilon^2-1)mG}{c^2\alpha^2}-\frac{mG}{\alpha r c^2}\left(4-\epsilon^2+1\right)+\frac{2mG}{c^2r^2}.$$

Therefore:

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}-\frac{2(\epsilon^2-1)mG}{c^2\alpha^2}-\frac{mG}{\alpha r c^2}\left(5-\epsilon^2\right)+\frac{2mG}{c^2r^2} \quad -(24)}$$

i.e.

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}+\frac{2mG}{c^2r^2}-\frac{mG}{\alpha r c^2}\left(5-\epsilon^2\right)+\frac{2(\epsilon^2-1)mG}{c^2\alpha^2} \quad -(25)$$

compared with the Einstein theory:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{\alpha}+\frac{3mG}{c^2r^2} \quad -(26)$$

If the assumption in eq. (6) is not used, eq. (25) becomes, with:

5)

$$MG = \frac{L_0^2}{\alpha m^2} \quad \text{— (27)}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} + \frac{2L_0^2}{m^2c^2r^2\alpha} - \frac{L_0^2(5-\epsilon^2)}{\alpha^2m^2c^2r} + \frac{2(\epsilon^2-1)L_0^2}{\alpha^3m^2c^2} \quad \text{— (28)}$$

By integrating eq. (28), a precessing ellipse will be found with a precession constant x that can be found graphically.

This method is valid only for:

$$x - 1 \sim 0 \quad \text{— (29)}$$

i.e.

$$v \ll c. \quad \text{— (30)}$$