239(5): Key Equations

The relativistic angular momentum is defined by:

$$L = \gamma m r^2 \frac{d\theta}{dt} = m r^2 \frac{d\theta}{d\tau} \quad -(1)$$

in which:

$$\boxed{L_0 = \frac{L}{\gamma} = \text{constant of motion}} \quad -(2)$$

Therefore:

$$\frac{d\theta}{dt} = \frac{L_0}{m r^2} = \left(\frac{L}{\gamma}\right)\frac{1}{m r^2} \quad -(3)$$

and

$$t = \int dt = \int \left(\frac{m r^2}{L_0}\right) d\theta \quad -(4)$$

i.e.

$$\boxed{t = \frac{m}{L_0}\int f^2(\theta)\, d\theta} \quad -(5)$$

Note carefully that L is not a constant, it is defined by:

$$L = L_0 \gamma \quad -(6)$$

where

$$L_0 = m r^2 \frac{d\theta}{dt} = \text{constant.} \quad -(7)$$

The time t in eq. (5) is the time in the

2) frame of reference in which the planet of mass m is moving.

The proper time τ is the time in which the planet of mass m is stationary with respect to a frame of reference fixed on the planet. So the time measured on the planet is τ.

Here:

$$\gamma = \frac{dt}{d\tau} \quad -(8)$$

So

$$\boxed{t = \gamma\tau} \quad -(9)$$

From eqs. (5) and (9):

$$\boxed{\tau = \frac{m}{L_0}\int \gamma f^2(\theta)\, d\theta} \quad -(10)$$

Therefore <u>plot</u> and <u>animate</u> eqs. <u>(5) and (10).</u>

The calculation of the relativistic force of Minkowski proceeds as follows. It is defined as:

$$\underline{F} = m\underline{a} = m\frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) \quad -(11)$$

where:

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) = \frac{d}{d\tau}\left(\gamma\frac{d\underline{r}}{dt}\right) = \gamma\frac{d}{dt}\left(\gamma\frac{d\underline{r}}{dt}\right) \quad -(12)$$

i.e.

$$\underline{a} = \gamma\left(\frac{d\gamma}{dt}\frac{d\underline{r}}{dt} + \gamma\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right) \quad -(13)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(14)$$

$$= \frac{dt}{d\tau}.$$

In eq. (14), v is defined by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(15)$$

In eq. (15):

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(16)$$

so:

$$v^2 = \left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(17)$$

From eq. (3):

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{L_0^2}{m^2 r^4} \quad -(18)$$

So:

$$\boxed{v^2 = \left(\frac{L_0}{mr^2}\right)^2\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right) \quad -(19)}$$

For an elliptical orbit:

$$r = \frac{d}{1+\epsilon\cos\theta} \quad -(20)$$

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{d}\sin\theta. \quad -(21)$$

so

$$v^2 = \left(\frac{L_0}{md}\right)^2\left(1+\epsilon^2+2\epsilon\cos\theta\right) \quad -(22)$$

for an elliptical orbit.

Note carefully that L_0 is used in the definition of v of the Lorentz factor.

Using the chain rule:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad -(23)$$

then $\underline{a}$ of eq. (13) becomes:

$$\underline{a} = \gamma^4 \frac{v}{c^2}\frac{dv}{dt}\frac{d\underline{r}}{dt} + \gamma^2\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) \quad -(24)$$

5) Therefore:

$$\underline{a} = \gamma \frac{d\gamma}{dt} \frac{d\underline{r}}{dt} + \gamma^2 \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) \quad -(25)$$

$$= \gamma \frac{d\gamma}{dt}\left(\frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r}\right) + \gamma^2\left(\frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})\right)$$

because the Coriolis acceleration is zero for all planar orbits.

It follows that:

$$\boxed{\underline{a} = \left(\gamma^2 \frac{d^2 r}{dt^2} + \gamma \frac{d\gamma}{dt}\frac{dr}{dt}\right)\underline{e}_r + \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \gamma \frac{d\gamma}{dt}\underline{\omega} \times \underline{r}} \quad -(26)$$

Using eq. (23):

$$\frac{d\gamma}{dt} = \frac{\gamma^3 v}{c^2}\frac{dv}{dt}, \quad -(27)$$

and

$$v = \frac{dr}{dt}, \quad \frac{dv}{dt} = \frac{d^2 r}{dt^2} \quad -(28)$$

So:

$$\underline{a} = \gamma^4 \frac{d^2 r}{dt^2} \underline{e}_r + \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \gamma \frac{d\gamma}{dt} \underline{\omega} \times \underline{r} \quad -(29)$$

Now use:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\frac{L_0^2}{m^2 r^3} \underline{e}_r \quad -(30)$$

and

$$\underline{\omega} \times \underline{r} = \omega r \, \underline{e}_\theta \quad -(31)$$

to find that:

$$\boxed{\underline{a} = \left(\gamma^4 \frac{d^2 r}{dt^2} - \frac{\gamma^2 L_0^2}{m^2 r^3} \right) \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \omega r \, \underline{e}_\theta}$$

$$-(32)$$

where

$$\underline{e}_r = \cos\theta \, \underline{i} + \sin\theta \, \underline{j} \quad -(33)$$

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(34)$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(35)$$

$$v^2 = \left(\frac{L_0}{mr^2}\right)^2 \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right) \quad -(36)$$