238(3): Relativistic Orbital Equation

In the non-relativistic theory the planar orbital equation is obtained from the acceleration of a planar orbit in plane polar coordinates:

$$\underline{a} = \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r \quad -(1)$$

$$= \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}).$$

This equation is transformed using the chain rules:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt}, \quad -(2)$$

and

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right) \frac{dr}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}. \quad -(3)$$

Therefore:

$$\frac{dr}{d\theta} = -r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(4)$$

and

$$\frac{dr}{dt} = -r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right) \frac{d\theta}{dt}, \quad -(5)$$

where

$$\frac{d\theta}{dt} = \frac{L}{mr^2}. \quad -(6)$$

So:

$$\boxed{\frac{dr}{dt} = -\frac{L}{m} \frac{d}{d\theta}\left(\frac{1}{r}\right)}. \quad -(7)$$

Similarly for any function f:

3) The relativistic counterpart of eq. (1) is:

$$\underline{a} = \left(\gamma^4 \frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \underline{\omega} \times \underline{r} \quad -(14)$$

where

$$\underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta . \quad -(15)$$

It would be very useful to transform eq. (14) into the format of eq. (12), thus making eq. (12) correctly relativistic. It would then be possible to calculate the relativistic force for any measured planar orbit.

In the relativistic theory:

$$L = m r^2 \frac{d\theta}{d\tau} = \gamma m r^2 \frac{d\theta}{dt} \quad -(16)$$

Therefore in eq. (5):

$$\frac{d\theta}{dt} = \frac{L}{\gamma m r^2} . \quad -(17)$$

So

$$\boxed{\frac{dr}{dt} = - \frac{L}{m\gamma} \frac{d}{d\theta} \left(\frac{1}{r} \right)} \quad -(18)$$

Similarly, eq. (8) becomes:

$$\frac{df}{dt} = \frac{df}{d\theta} \frac{d\theta}{dt} = \frac{L}{\gamma m r^2} \frac{df}{d\theta} \quad -(19)$$

2)

$$\frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{df}{d\theta}. \quad -(8)$$

If

$$f = \frac{dr}{dt} \quad -(9)$$

then

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d^2r}{dt^2} = \frac{L}{mr^2}\frac{d}{d\theta}\left(-\frac{L}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right)\right) \quad -(10)$$

i.e

$$\boxed{\frac{d^2r}{dt^2} = -\left(\frac{L}{mr}\right)^2\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)} \quad -(11)$$

With the transformation (11) eq. (1) becomes:

$$\boxed{\underline{a} = -\left(\frac{L}{mr}\right)^2\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r}. \quad -(12)$$

Note carefully that eq. (12) is valid for any planar orbit, and that it includes the centripetal acceleration. The centripetal acceleration is the same for any planar orbit, and the Coriolis acceleration is zero for any planar orbit.

Eq. (12) is much more useful than eq. (1) because it gives the force for any planar orbit:

$$\underline{F} = m\,\underline{a}. \quad -(13)$$

4)

so eq. (11) becomes:

$$\boxed{\frac{d^2 r}{dt^2} = -\left(\frac{L}{\gamma m r}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)} \quad -(20)$$

Therefore using eq. (20) in eq. (14):

$$\underline{a} = -\left(\left(\frac{\gamma L}{m r}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{L^2}{m^2 r^3}\right)\underline{e}_r + \frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2}\,\omega r\,\underline{e}_\theta \quad -(21)$$

Using eqs. (17), (18) and (20):

$$\boxed{\begin{aligned}\underline{a} = &-\left(\frac{L}{m r}\right)^2\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \\ &+ \frac{L^4}{(m^2 c)^2 r^3}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta\end{aligned}} \quad -(22)$$

and the relativistic force is:

$$\underline{F} = m\,\underline{a} \quad -(23)$$

for any orbit:

$$r = f(\theta) \quad -(24)$$

QED