238(8): Time Evolution of Orbits

In the non-relativistic theory the time evolution of orbits is calculated from the conserved angular momentum:

$$L_0 = mr^2 \frac{d\theta}{dt} \quad -(1)$$

$$= \text{constant}.$$

Therefore:

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2} \quad -(2)$$

and the planar orbit in general is:

$$r = f(\theta). \quad -(3)$$

Therefore

$$dt = \frac{m}{L_0} f^2(\theta)\, d\theta. \quad -(4)$$

The time taken for a revolution of 2π radians is:

$$t = \int dt = \frac{m}{L_0} \int_0^{2\pi} f^2(\theta)\, d\theta \quad -(5)$$

For an elliptical orbit:

$$r = f(\theta) = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(6)$$

and for a hyperbolic spiral orbit:

$$r = f(\theta) = -\frac{r_0}{\theta}. \quad -(7)$$

The hyperbolic spiral orbit is much easier to deal with

2)

analytically, but computer algebra can deal with any function $f(\theta)$. In the case of the hyperbolic spiral in the non-relativistic approximation:

$$\frac{d\theta}{dt} = \frac{dr}{dt}\frac{d\theta}{dr} = \frac{L_0}{mr^2} \quad -(8)$$

so

$$\frac{dr}{dt} = \frac{L_0}{mr^2}\frac{dr}{d\theta} = -\frac{L_0}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(9)$$

i.e.

$$\frac{dr}{dt} = -\frac{L_0}{m}\frac{d}{d\theta}\left(-\frac{\theta}{r_0}\right) = \frac{L_0}{mr_0}. \quad -(10)$$

Therefore

$$\boxed{r = \frac{L_0}{mr_0}t}. \quad -(11)$$

This is a simple, linear dependence of r on t, which shows that the distance of a star from the centre of the galaxy increases with time t in the observer frame.

The relativistic equivalent of eq. (8) is:

$$\frac{d\theta}{d\tau} = \frac{L_0}{mr^2}. \quad -(12)$$

where

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(13)$$

and:

3)

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(14)$$

$$= \left(\frac{dr}{dt}\right)^2\left(1 + r^2\left(\frac{d\theta}{dr}\right)^2\right)$$

In this equation:

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{L_0}{mr_0}\right)^2 \quad -(15)$$

and

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{r_0^2}{r^4} \quad -(16)$$

So

$$\boxed{v^2 = \left(\frac{L_0}{mr_0}\right)^2\left(1 + \frac{r_0^2}{r^2}\right)} \quad -(17)$$

From eq. (17)

$$\boxed{v \xrightarrow[r \to \infty]{} \frac{L_0}{mr_0} = \text{constant}} \quad -(18)$$

as observed experimentally.

Therefore:

$$\gamma = \left(1 - \left(\frac{L_0}{mr_0c}\right)^2\left(1 + \left(\frac{r_0}{r}\right)^2\right)\right)^{-1/2} \quad -(19)$$

4) Therefore:

$$\frac{d\theta}{d\tau} = \frac{dr}{d\tau}\frac{d\theta}{dr} = \frac{L_0}{mr^2} \qquad -(20)$$

and

$$\frac{dr}{d\tau} = \frac{L_0}{mr^2}\frac{dr}{d\theta} = \frac{L_0}{mr_0} \qquad -(21)$$

and

$$\boxed{r = \frac{L_0}{mr_0}\tau} \qquad -(22)$$

In the frame of the proper time τ, r is again simply proportional to τ. The proper time is the time measured in the frame in which the particle is at rest. The observer time t in the lab. frame is the time in the frame in which the particle is moving.

By definition of the Lorentz transformation:

$$\gamma = \frac{dt}{d\tau} \qquad -(23)$$

so

$$\tau = \frac{t}{\gamma} \qquad -(24)$$

i.e.

$$\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} t \qquad -(25)$$

therefore from eqs. (19), (22) and (25):

5)

$$\boxed{r = \left(1 - \left(\frac{L_0}{m r_0 c}\right)^2 \left(1 + \left(\frac{r_0}{r}\right)^2\right)\right)^{1/2} \frac{L_0}{m r_0} t}$$

$$-(26)$$

This is no longer a simple linear dependence of r on t. It is convenient to invert eq. (26) to give:

$$\boxed{t = \frac{m r_0}{L_0} r \left(1 - \left(\frac{L_0}{m r_0 c}\right)^2 \left(1 + \left(\frac{r_0}{r}\right)^2\right)\right)^{-1/2}} \quad -(27)$$

and plot t against r. The time taken for the star to move a distance r can then be found. In this relativistic theory of whirlpool galaxies the stars can move with speeds approaching c.