238(2): Development of the Relativistic Acceleration in Plane Polar Coordinates

In general the relativistic acceleration is defined by:

$$\underline{a} = \gamma \left(\frac{d\gamma}{dt} \frac{d\underline{r}}{dt} + \gamma \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right), \quad -(1)$$

and the Minkowski force is:

$$\underline{F} = m\,\underline{a}. \quad -(2)$$

Using the chain rule:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad -(3)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

and

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(5)$$

Therefore:

$$\frac{d\gamma}{dv} = \frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(6)$$

$$= \gamma^3 v / c^2$$

and

$$\underline{a} = \gamma^4 \frac{v}{c^2}\frac{dv}{dt}\frac{d\underline{r}}{dt} + \gamma^2 \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) \quad -(7)$$

2)

In plane polar coordinates:

$$\underline{a} = \left(\frac{d\gamma}{d\tau} \frac{dr}{dt} + \gamma \frac{d^2 r}{dt^2} \right) \underline{e}_r + \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$+ \frac{d\gamma}{d\tau} \underline{\omega} \times \underline{r} + \gamma^2 \left(\frac{d\underline{\omega}}{dt} \times \underline{r} + 2 \underline{\omega} \times \frac{dr}{dt} \underline{e}_r \right) \quad -(8)$$

In Cartesian coordinates:

$$\underline{a} = \frac{d}{d\tau} \left(\gamma \frac{d\underline{r}}{dt} \right) = \gamma \frac{d}{dt} \left(\gamma \frac{d\underline{r}}{dt} \right) \quad -(9)$$

$$= \gamma \frac{d\gamma}{dt} \frac{d\underline{r}}{dt} + \gamma^2 \frac{d}{dt} \frac{d\underline{r}}{dt}$$

with axes fixed. So:

$$\underline{a} \, (\text{Cartesian}) = \left(\gamma \frac{d\gamma}{dt} \frac{dr}{dt} + \gamma^2 \frac{d^2 r}{dt^2} \right) \underline{e}_r \quad -(10)$$

in which:

$$v = \frac{dr}{dt}, \quad \frac{d^2 r}{dt^2} = \frac{dv}{dt}, \quad -(11)$$

$$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2}, \quad \frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2}.$$

$$\text{so:} \quad \frac{d\gamma}{dt} = \frac{d\gamma}{dv} \frac{dv}{dt} = \frac{\gamma^3 v}{c^2} \frac{dv}{dt}. \quad -(12)$$

3) Therefore:

$$\underline{a}\left(\text{Cartesian}\right) = \left(\gamma^4 \frac{v^2}{c^2} + \gamma^2\right) \frac{dv}{dt}\, \underline{e}_r, \quad -(13)$$

in which:

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}, \quad -(14)$$

so:

$$\boxed{\underline{a}\left(\text{Cartesian}\right) = \gamma^4 \frac{d^2 r}{dt^2}\, \underline{e}_r} \quad -(15)$$

From eq. (15) & eq. (8):

$$\underline{a}\left(\text{plane polar}\right) = \gamma^4 \frac{d^2 r}{dt^2}\, \underline{e}_r + \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underset{\sim}{r})$$
$$+ \frac{d\gamma}{d\tau}\, \underline{\omega} \times \underline{r} + \gamma^2 \left(\frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\, \underline{e}_r \right) \quad -(16)$$

For planar orbits:

$$\frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\, \underline{e}_r = \underline{0} \quad -(17)$$

i.e. the relativistic Coriolis acceleration vanishes for all planar orbits.

4) This result follows from the fact that the relativistic Coriolis acceleration is:

$$\underline{a}\,(\text{Coriolis}) = \left(r \frac{d}{d\tau}\frac{d\theta}{d\tau} + 2\frac{dr}{d\tau}\frac{d\theta}{d\tau} \right) \underline{e}_\theta \quad -(18)$$

where

$$L = mr^2 \frac{d\theta}{d\tau} = \text{constant} \quad -(19)$$

is the conserved total angular momentum. Therefore:

$$\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{L}{mr^2}\right) = \frac{d}{dr}\left(\frac{L}{mr^2}\right)\frac{dr}{d\tau}$$

$$= -\frac{2L}{mr^3}\frac{dr}{d\tau} \quad -(20)$$

So

$$\underline{a}\,(\text{Coriolis}) = \left(-\frac{2L}{mr^2}\frac{dr}{d\tau} + \frac{2L}{mr^2}\frac{dr}{d\tau} \right) \underline{e}_\theta \quad -(21)$$

$$= \underline{0}$$

QED. Therefore for all planar orbits:

$$\boxed{\underline{a} = \gamma^4 \frac{d^2 r}{dt^2}\underline{e}_r + \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\gamma}{d\tau}\,\underline{\omega} \times \underline{r}} \quad -(22)$$

5) Here:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \underline{e}_r, \quad -(23)$$

$$\underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta, \quad -(24)$$

$$L = \gamma m r^2 \omega, \quad -(25)$$

and

$$\frac{d\gamma}{d\tau} = \gamma \frac{d\gamma}{dt} = \gamma \frac{d\gamma}{dv} \frac{dv}{dt}$$

$$= \gamma^4 \frac{v}{c^2} \frac{dv}{dt}. \quad -(26)$$

$$= \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2}$$

Therefore:

$$\underline{a} = \gamma^4 \frac{d^2 r}{dt^2} \underline{e}_r - \frac{L^2}{m^2 r^3} \underline{e}_r \quad -(27)$$

$$+ \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \omega r \underline{e}_\theta$$

The relativistic force experienced in the Cartesian frame is found from eq. (27) as follows:

$$\underline{F} = \gamma^4 m \frac{d^2 r}{dt^2} \underline{e}_r \quad (\text{Cartesian})$$

$$= m\underline{a} + \frac{L^2}{mr^3}\underline{e}_r - \frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2} m\omega r \underline{e}_\theta \quad -(28)$$

It contains a term in $\underline{e}_\theta$ which is not present in the classical result. This is the normally directed force:

$$\boxed{\underline{F}(\text{normal}) = -\frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2} m \underline{\omega} \times \underline{r}} \quad -(29)$$

where

$$\underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta \quad -(30)$$

and

$$\underline{e}_\theta = -\sin\theta\, \underline{i} + \cos\theta\, \underline{j} \quad -(31)$$

Eq. (29) is a purely relativistic result that does not exist in classical orbital theory. To an observer in the static frame:

$$\boxed{\underline{F} = \left(\gamma^4 m \frac{d^2 r}{dt^2} - \frac{L^2}{mr^3}\right)\underline{e}_r + \left(\frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2}\right) m\omega r \underline{e}_\theta} \quad -(32)$$