237(8) : Development of Kinematics with Cartan Geometry.

The Cartan covariant derivative is defined as:

$$\frac{D\underline{V}}{dt} = \left(\frac{d\underline{V}}{dt}\right)_{\substack{\text{axes}\\ \text{fixed}}} + \underline{\omega} \times \underline{V} \quad -(1)$$

where

$$\underline{V} = V\underline{e}_r \quad -(2)$$

for simplicity of development.

The Velocity

$$\underline{v} = \frac{D\underline{r}}{dt} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad -(3)$$

where

$$\frac{d\underline{r}}{dt} := \left(\frac{d\underline{r}}{dt}\right)_{\substack{\text{axes}\\ \text{fixed}}} \quad -(4)$$

By definition:

$$\frac{D\underline{r}}{dt} = \frac{D(r\underline{e}_r)}{dt}$$

$$= \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(5)$$

So:

$$\left(\frac{d\underline{r}}{dt}\right)_{\substack{\text{axes}\\ \text{fixed}}} = \frac{dr}{dt}\underline{e}_r \quad -(6)$$

and

$$\underline{\omega} \times \underline{r} = r\frac{d\underline{e}_r}{dt} \quad -(7)$$

2)

The Acceleration

This is defined as:

$$\underline{a} = \frac{D\underline{v}}{dt} = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} \qquad -(8)$$

where

$$\frac{d\underline{v}}{dt} := \left(\frac{d\underline{v}}{dt}\right)_{\text{axes fixed}} \qquad -(9)$$

From fundamental kinematics:

$$\underline{a} = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} = \left(\ddot{r} - r\omega^2\right)\underline{e}_r + \left(r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega\right)\underline{e}_\theta$$

$$= \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) + r\frac{d\omega}{dt}\underline{e}_\theta + 2\frac{dr}{dt}\omega\underline{e}_\theta \qquad -(10)$$

with:

$$\underline{e}_r \times \underline{e}_\theta = \underline{k}$$
$$\underline{k} \times \underline{e}_r = \underline{e}_\theta \qquad -(11)$$
$$\underline{e}_\theta \times \underline{k} = \underline{e}_r$$

So:

$$\frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} = \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) + r\frac{d\omega}{dt}\left(\underline{k} \times \underline{e}_r\right) + 2\frac{dr}{dt}\omega\left(\underline{k} \times \underline{e}_r\right)$$

$$= \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \left(\frac{dr}{dt}\underline{e}_r\right) \qquad -(12)$$

3)

From eq. (3):

$$\underline{v} = \frac{dr}{dt} + \underline{\omega} \times \underline{r} \quad -(13)$$

so in eq. (8):

$$\underline{a} = \frac{d}{dt}\left(\frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r}\right)_{\text{axes fixed}} + \underline{\omega} \times \left(\frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r}\right) \quad -(14)$$

$$= \frac{d^2 r}{dt^2}\underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{\substack{\text{axes}\\\text{fixed}}} + \underline{\omega} \times \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(15)$$

In this equation:

$$\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{\substack{\text{axes}\\\text{fixed}}} = \underline{\omega} \times \frac{dr}{dt}\underline{e}_r$$

So:

$$\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \left(\frac{dr}{dt}\underline{e}_r\right) \quad -(16)$$

which is eq. (12), QED.

Therefore all planar orbits and all types

4) of rotational motion are described by the covariant derivative of type (1). In general relativity eq. (1) becomes:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \quad -(17)$$

where $\omega^a_{\mu b}$ is the Cartan spin connection.

This type of analysis was applied in UFT55 using the torsion and first Cartan structure equation:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(18)$$

Eq. (18) can be used to define the gravitational field, in which the tetrad defines the spin connection. The Cartan identity:

$$D \wedge T := R \wedge q \quad -(19)$$

gives four vector field equations.

The new insight of kinematics is that the spin connection is a generalization of eq. (1), in which the spin connection is the angular velocity vector.