237(2): Kinematics of the Elliptical Trajectory.

The velocity of the elliptical trajectory is given by:

$$\underline{v} = \left(\frac{L}{m}\right)\left(\frac{1}{r}\,\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right). \quad -(1)$$

Note carefully that this is the result of pure kinematics:

$$\underline{v} = \frac{d\underline{r}}{dt}, \quad \underline{L} = \underline{r} \times \underline{p} = m\,\underline{r} \times \underline{v}. \quad -(2)$$

The total velocity is:

$$\underline{v} = \frac{dr}{dt}\,\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(3)$$

where

$$\underline{\omega} = \frac{d\theta}{dt}\,\underline{k} \quad -(4)$$

is the spin connection and angular velocity vector.

The trajectory of the ellipse and conical sections is:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(5)$$

where α is the half right latitude and ϵ the eccentricity.

The velocity $\underline{v}$ is that of a particle of mass m moving on an ellipse. The angular momentum $\underline{L}$ is a constant of motion:

$$\begin{aligned}\underline{L} &= m\,\underline{r} \times \underline{v} = m\left(\underline{r} \times \left(\frac{dr}{dt}\,\underline{e}_r + \omega r\,\underline{e}_\theta\right)\right) \\ &= m r\,\underline{e}_r \times \left(\frac{dr}{dt}\,\underline{e}_r + \omega r\,\underline{e}_\theta\right) \\ &= m r^2 \omega\,\underline{e}_r \times \underline{e}_\theta = m r^2 \omega\,\underline{k} \quad -(6)\end{aligned}$$

2)

Note carefully that the velocity case evaluated without any reference to Newtonian theory. This is because fundamental kinematics are more fundamental and more general than Newtonian theory.

From eq. (5):

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\sin\theta \quad -(7)$$

So

$$v^2 = \left(\frac{L}{m}\right)^2\left(\frac{1}{r^2} + \left(\frac{\epsilon}{\alpha}\right)^2 \sin^2\theta\right) \quad -(8)$$

Using eq. (6):

$$v^2 = \omega^2 r^2\left(1 + \left(\frac{\epsilon r}{\alpha}\right)^2 \sin^2\theta\right) \quad -(9)$$

which is eq. (8) of note 236(5), QED. As in that note:

$$v^2 = \frac{1}{\alpha}\left(\frac{L}{m}\right)^2\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(10)$$

where the semi major axis of the ellipse is:

$$a = \frac{\alpha}{1-\epsilon^2} \quad - (11)$$

Eq. (10) gives the velocity of a mass m moving along an ellipse.

The transition to Newtonian theory is made with

$$\alpha = \frac{L^2}{m^2 MG} \quad -(12)$$

so:

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(13)$$

From very general and fundamental kinematics the motion of the mass m along an ellipse produces the force:

$$\underline{F} = m\underline{a} = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \quad -(14)$$

From eq. (5):

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\theta = \frac{1}{\alpha} - \frac{1}{r} \quad -(15)$$

So

$$\underline{F} = -\frac{L^2}{mr^2\alpha}\underline{e}_r \quad -(16)$$

This result is due completely to the fact that the acceleration of a mass m moving on an ellipse or conical section is:

$$\boxed{\begin{aligned}\underline{a} = \frac{d^2\underline{r}}{dt^2} &= \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega}\times(\underline{\omega}\times\underline{r}) \\ &= -\frac{1}{\alpha}\left(\frac{L}{mr}\right)^2\underline{e}_r \quad -(17)\end{aligned}}$$

The Newtonian theory assumes eq. (12) and gives:

$$\underline{F} = -\frac{mMG}{r^2}\underline{e}_r \quad -(18)$$

known as "the inverse square law of universal gravitation".

4)

In the Newtonian theory therefore:

$$\underline{a} = -\frac{MG}{r^2}\underline{e}_r = \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega}\times(\underline{\omega}\times\underline{r})$$

$$= \underline{g} \qquad -(19)$$

where $\underline{g}$ is "the acceleration due to gravity". For the earth, M is the mass of the earth and G is Newton's constant. At the surface of the earth, r is the radius of the earth.

Note carefully that the Newtonian theory is a way of expressing the more general result (17). The Newtonian theory happens to work in the solar system, but in galaxies fails completely. Also note carefully that the acceleration due to gravity $\underline{g}$ includes the centrifugal acceleration $\underline{\omega}\times(\underline{\omega}\times\underline{r})$.

The equivalence principle assumes that:

$$\underline{F} = m\underline{a} = -\frac{mMG}{r^2}\underline{e}_r \qquad -(20)$$

in Newtonian theory.

The general kinematics of note 237(1) give the potential energy:

$$U = \int \frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) dr \qquad -(21)$$

5) For an elliptical trajectory, or any trajectory on a conical section, eqs. (15) and (21) give:

$$U = \frac{L^2}{m\alpha}\int\frac{dr}{r^2} = -\left(\frac{L^2}{m\alpha}\right)\frac{1}{r} \quad -(22)$$

$$= -\int F\,dr$$

The Newtonian result is given by eq. (12) and is:

$$U = -\frac{mMG}{r} \quad -(23)$$

This is known as "the gravitational potential energy".

The potential energy is defined as:

$$F = -\frac{\partial U}{\partial r} \quad -(24)$$

The total energy, or Hamiltonian, is defined as:

$$H = T + U \quad -(25)$$

$$= E$$

where T is the kinetic energy:

$$T = \frac{1}{2}mv^2, \quad -(26)$$

so

$$E = \frac{1}{2}mv^2 + U \quad -(27)$$

In the general kinematics:

$$E = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) - \int F \, dr \quad -(28)$$

The kinetic energy in general is defined by the work done:

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{r} = T_2 - T_1 . \quad -(29)$$

If

$$T_1 = 0 \quad -(30)$$

then

$$W = T = \left(\int \underline{F} \cdot d\underline{r} \right)_{kinetic} \quad -(31)$$

So the total energy is:

$$\boxed{E = \left(\int \underline{F} \cdot d\underline{r} \right)_{kinetic} - \left(\int F \, dr \right)_{potential} = H = \text{constant} \quad -(32)}$$

In other words the total energy is due entirely to the integrals over the acceleration $\underline{a}$. Again the fundamental kinematic theory is more general than the Newtonian theory.

The kinetic energy is:

$$T = W = \int \frac{d}{dt} (m \underline{v}) \cdot d\underline{r} \quad -(33)$$

where

$$d\underline{r} = \underline{v} \, dt \quad -(34)$$

7) So
$$T = W = \int \left(\frac{d}{dt} m \underline{v} \right) \cdot \underline{v} \, dt \quad - (35)$$
$$= \frac{m}{2} \int \frac{d}{dt} (\underline{v} \cdot \underline{v}) \, dt$$
$$= \frac{1}{2} m v^2$$

The potential energy is defined as:
$$W = U_1 - U_2 = \int_1^2 \underline{F} \cdot \underline{dr} \quad - (36)$$

and is the work done in moving m from 1 to 2.

Eq. (36) implies that:
$$\underline{F} = - \underline{\nabla} U \quad - (37)$$

so:
$$\int_1^2 \underline{F} \cdot \underline{dr} = - \int_1^2 (\underline{\nabla} U) \cdot \underline{dr}$$
$$= - \int_1^2 dU = U_1 - U_2 \quad - (38)$$

QED. - It is seen that the definitions of kinetic and potential energy result from the equivalence principle:

8)

$$\boxed{\underline{F} = m\frac{d\underline{v}}{dt} = -\underline{\nabla} U} \quad -(39)$$

So in general kinematics:

$$H = E = \left(\int \underline{F}\cdot\underline{dr}\right)_{kinetic} + \left(\int \underline{F}\cdot\underline{dr}\right)_{potential} \quad -(40)$$

where:

$$\left(\int \underline{F}\cdot\underline{dr}\right)_{kinetic} = \int m\frac{d\underline{v}}{dt}\cdot\underline{dr} \quad -(41)$$

$$\left(\int \underline{F}\cdot\underline{dr}\right)_{potential} = -\int \underline{\nabla} u\cdot\underline{dr} \quad -(42)$$

For the elliptical trajectory:

$$E = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{L^2}{m\alpha r} \quad -(43)$$

and again this is the result of pure kinematics.
It can be shown that eq. (43) is the equation of an ellipse. So the Newtonian theory is a particular choice of ϵ and α.

9) Use in eq. (43):

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \omega \frac{dr}{d\theta} \quad -(44)$$

and

$$L = m\omega^2 r \quad -(45)$$

to find that:

$$E = \frac{1}{2} m\omega^2 \left(\left(\frac{dr}{d\theta}\right)^2 + r^2 \right) - \frac{L^2}{m\alpha r} \quad -(46)$$

$$= \frac{1}{2}\frac{L^2}{mr^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2 \right) - \frac{L^2}{m\alpha r}$$

The ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(47)$$

so

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{\epsilon r^2}{\alpha}\right)^2 \sin^2\theta$$

$$= \left(\frac{\epsilon r^2}{\alpha}\right)^2 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)$$

$$= \left(\frac{\epsilon r^2}{\alpha}\right)^2 - \frac{r^4}{\alpha^2}\left(\frac{\alpha^2}{r^2} - 2\frac{\alpha}{r} + 1\right)$$

$$= \frac{r^4}{\alpha^2}\left(\epsilon^2 - 1\right) + 2r^2\left(\frac{r}{\alpha}\right) - r^2 \quad -(48)$$

Using eq. (48) in eq. (46):

$$E = \frac{L^2}{m}\left[\frac{1}{2r^4}\left(\frac{r^4}{\alpha^2}(\epsilon^2-1)+\frac{2r^3}{\alpha}\right)-\frac{1}{\alpha r}\right]$$

$$= \frac{L^2}{m\alpha}\left[\frac{(\epsilon^2-1)}{2\alpha}+\frac{1}{\alpha r}-\frac{1}{\alpha r}\right\}$$

$$= (\epsilon^2-1)\frac{L^2}{2m\alpha^2} \quad -(49)$$

So eqs. (46) and (48) are sol ellipses if:

$$\boxed{\frac{\epsilon^2-1}{\alpha^2} = \frac{2mE}{L}} \quad -(50)$$

The total energy is therefore:

$$E = T + U = \frac{L}{2m}\left(\frac{\epsilon^2-1}{\alpha^2}\right)$$

$$= \text{constant} \quad -(51)$$

Note that the total energy is due entirely to the spin connection ω, because:

$$L = mr^2\omega = \text{constant} \quad -(52)$$

Note carefully that the result (50) is a general result valid for the motion of an

11) object m on an elliptical trajectory, or that of any conical section. The Newtonian theory asserts eq. (12).

Finally note carefully that all the quantities E, T, U, W and H can be obtained from the acceleration for any orbit in a plane. This produces a general theory of planar orbits in terms of the spin connection of Cartan and ECE theory.