237(1): Summary of Results and New Equation of Orbital Velocity in a Plane

In general the acceleration associated with planar rotation is:

$$\underline{a} = \frac{d^2\underline{r}}{dt^2} = \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{d\underline{r}}{dt} \quad -(1)$$

The Coriolis acceleration is:

$$\underline{a}_{Coriolis} = \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{d\underline{r}}{dt} \quad -(2)$$

$$= \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right) \underline{e}_\theta .$$

This is zero for any planar rotation because:

$$\underline{L} = \underline{r} \times \underline{p} = mr^2 \frac{d\theta}{dt} \underline{k} \quad -(3)$$

and

$$\frac{d\theta}{dt} = \frac{L}{mr^2} . \quad -(4)$$

So

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{dt}\left(\frac{L}{mr^2}\right)$$

$$= \frac{dr}{dt}\frac{d}{dr}\left(\frac{L}{mr^2}\right) \quad -(4)$$

$$= -\frac{2L}{mr^3}\frac{dr}{dt}$$

2)

So:

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{2L}{mr^2}\frac{dr}{dt} + 2\frac{dr}{dt}\frac{L}{mr^2}$$

$$= 0. \qquad -(5)$$

QED.

Therefore for any planar orbit:

$$\underline{a} = \frac{d^2r}{dt^2}\underline{e}_r + \underline{\omega}\times(\underline{\omega}\times\underline{r}). \quad -(6)$$

In this equation:

$$f := \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad -(7)$$

and

$$\frac{df}{dt} = \frac{df}{dr}\frac{dr}{dt} = f\frac{df}{dr} \quad -(8)$$

So:

$$\frac{d^2r}{dt^2} = \left(\frac{L}{mr}\right)^2\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right). \quad -(9)$$

Now note that:

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = \frac{d\theta}{dr}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) \quad -(10)$$

so

$$\frac{d^2r}{dt^2} = \left(\frac{L}{mr}\right)^2\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right). \quad -(11)$$

Next note that:

3)

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \quad -(12)$$

so

$$\frac{d^2 r}{dt^2} = -\left(\frac{L}{mr}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(13)$$

Therefore for any planar orbit:

$$\underline{a} = -\left(\frac{L}{mr}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(14)$$

in which

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \, \underline{e}_r = -\frac{L^2}{mr^3}\underline{e}_r \quad -(15)$$

So:

$$\boxed{\underline{a} = -\left(\frac{L}{mr}\right)^2 \left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r} \quad -(16)$$

The acceleration of any planar orbit is determined entirely by the spin connection:

$$\underline{\omega} = \omega \underline{k} = \frac{L}{mr^2}\underline{k} \quad -(17)$$

and $\underline{a}$ is always directed along $-\underline{e}_r$. This is a result of pure kinematics.

4) At this point, force is defined as:

$$\underline{F} = m\underline{a} \quad -(18)$$

So

$$\boxed{\underline{F} = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r} \quad -(19)$$

for any planar orbit.

Lagrangian dynamics also give eq. (19), and are more general than Newtonian dynamics.

The orbital velocity is given by:

$$\underline{v} = \frac{dr}{dt}\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(20)$$

$$= \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r}$$

From eq. (12):

$$\frac{dr}{d\theta} = -r^2\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{dr}{dt}\frac{dt}{d\theta}. \quad -(21)$$

So

$$\frac{dr}{dt} = -\omega r^2\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(22)$$

It follows that:

$$\underline{v} = -\omega r^2\frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r + \omega r\,\underline{e}_\theta$$

$$-(23)$$

5) i.e. for any planar orbit:

$$\underline{v} = \frac{L}{mr}\,\underline{e}_\theta - \omega r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r \qquad -(24)$$

$$\boxed{\underline{v} = \left(\frac{L}{m}\right)\left(\frac{1}{r}\,\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right)} \qquad -(25)$$

It follows that:

$$v^2 = \left(\frac{L}{m}\right)^2\left[\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right] \qquad -(26)$$

for any planar orbit.

The kinetic energy of any planar orbit is:

$$T = \frac{1}{2}mv^2 = \frac{L^2}{2m}\left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right) \qquad -(27)$$

The potential energy of any planar orbit is

given by: $$F = -\frac{\partial U}{\partial r} \qquad -(28)$$

From eqs. (19) and (28):

$$\boxed{U = \int \frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) dr} \qquad -(29)$$

6) The Hamiltonian is :

$$H = E = T + U \quad -(30)$$

and the Lagrangian is :

$$\mathcal{L} = T - U. \quad -(31)$$

The limiting velocity is :

$$\underline{v} \xrightarrow[r \to \infty]{} -\frac{L}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r \quad -(32)$$

or

$$v = v_\infty \xrightarrow[r \to \infty]{} -\frac{L}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(33)$$

i.e

$$\boxed{\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{m v_\infty}{L}} \quad -(34)$$

for any planar orbit.

For a whirlpool galaxy :

$$v_\infty = \text{constant} \neq 0 \quad -(35)$$

so

$$\boxed{\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r_0}} \quad -(36)$$

where

$$r_0 = \frac{L}{m v_\infty} \quad -(37)$$

Therefore :

7)

$$\boxed{r = -\frac{r_0}{\theta}} \quad -(38)$$

and the orbit is a hyperbolic spiral as desired QED

From eqs. (19) and (38):

$$\boxed{\underline{F} = -\frac{L^2}{mr^3}\,\underline{e}_r} \quad -(39)$$

and the force law is inverse cube in r.

Eq. (39) means that the force needed to keep a star in a hyperbolic spiral orbit is inverse cube in r and directed towards the centre of the galaxy.

In Newtonian dynamics:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad -(40)$$

so

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\sin\theta, \quad -(41)$$

$$-1 \leqslant \sin\theta \leqslant 1 \quad -(42)$$

and

$$\alpha = a\left(1-\epsilon^2\right) \quad -(43)$$

where a is the semi major axis.

8)

Therefore:

$$v_\phi \left(\text{Newton}\right) = -\frac{L}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right)$$

$$= \frac{\epsilon L \sin\theta}{\alpha m} \qquad -(44)$$

$$= \frac{\epsilon L \sin\theta}{a(1-\epsilon^2)m} \longrightarrow 0$$

because:

$$a \xrightarrow[r \to \infty]{} \infty \qquad -(45)$$

and

$$-1 \leq \sin\theta \leq 1, \qquad -(46)$$

a result that confirms that the Newtonian velocity vanishes at infinite r, as shown in another way in UFT236.