237(7): The Acceleration due to Gravity

In planar orbits the acceleration is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(1)$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\underset{\sim}{T} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{N} \quad -(2)$$

where

$$\underset{\sim}{T} = \underline{e}_r \quad \text{and} \quad \underline{N} = \underline{e}_\theta \quad -(3)$$

are the tangent and normal unit vectors of the Frenet frame of reference. The Frenet binormal vector is:

$$\underline{B} = \underline{k} = \underline{T} \times \underline{N} = \underline{e}_r \times \underline{e}_\theta \quad -(4)$$

The Frenet curvature is defined by:

$$\boxed{\rho = \frac{dr}{d\theta} = \frac{v}{\omega} = \frac{dr}{dt}\frac{dt}{d\theta}} \quad -(5)$$

In all planar orbits:

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta = \underline{0} \quad -(6)$$

i.e. the Coriolis acceleration vanishes. So for the planar orbits:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r$$

$$= \left(\frac{d^2r}{dt^2} - \omega^2 r\right)\underline{e}_r \quad -(7)$$

$$\boxed{\underline{a} = \frac{d^2r}{dt^2}\,\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})} \quad -(8)$$

2) The acceleration due to gravity is:

$$\underline{g} = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(9)$$

the centripetal acceleration is:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \, \underline{e}_r \quad -(10)$$

In an elliptical orbit:

$$\underline{a} = -\frac{L^2}{m r^2 \alpha} \underline{e}_r \quad -(11)$$

where

$$L = |\underline{L}| = |\underline{r} \times \underline{p}| = m r^2 \omega \quad -(12)$$

is the constant angular momentum magnitude. The ellipse is defined by

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(13)$$

where α is the half right magnitude (semi latus rectum) and ϵ the eccentricity. So for an elliptical orbit:

$$\underline{g} = -\frac{L^2}{m^2 r^2 \alpha} \underline{e}_r \quad -(14)$$

If

$$\alpha = \frac{L^2}{m^2 M G} \quad -(15)$$

this becomes

$$\underline{g} = -\frac{MG}{r^2} \underline{e}_r \quad -(16)$$

From eq. (9):

$$\frac{d^2 r}{dt^2}\underline{e}_r = \underline{g} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(17)$$

The centrifugal acceleration is:

$$-\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \omega^2 r \underline{e}_r . \quad -(18)$$

In an elliptical orbit:

$$\frac{d^2 r}{dt^2} = -\frac{L^2}{m^2 r^2 \alpha} + \omega^2 r$$

$$= -\frac{L^2}{m^2 r^2 \alpha} + \frac{L^2}{m^2 r^3} \quad -(19)$$

i.e.

$$\boxed{\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \left(\frac{1}{r} - \frac{1}{\alpha}\right)} \quad -(20)$$

This is a purely kinematic result. The Newtonian theory uses eq. (15) to assert:

$$\frac{d^2 r}{dt^2} = -\frac{MG}{r^2} + \frac{L^2}{m^2 r^3} \quad -(21)$$

The Newtonian force is:

$$F = m\frac{d^2 r}{dt^2} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} \quad -(22)$$

4) However, the total force is:

$$\underline{F} = m\underline{g} = \underline{F}_{Newtonian} + m\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= -\frac{mMG}{r^2}\,\underline{e}_r \quad -(23)$$

Eq. (23) is often described as:

$$\underline{F}_{lab} = \underline{F}_{rotating} + m\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= -\frac{mMG}{r^2}\,\underline{e}_r \quad -(24)$$

so

$$\underline{F}_{rotating} = \underline{F}_{lab} - m\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= \left(-\frac{mMG}{r^2} + \frac{L^2}{mr^3}\right)\underline{e}_r \quad -(25)$$

The force as experienced on planet earth is eq. (25) and as experienced on the sun is eq. (24).

Eqs. (24) and (25) could be written as:

$$\underline{F}_{Sun} = \underline{F}_{Earth} + m\underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(26)$$

5)

and:

$$\underline{F}_{earth} = \underline{F}_{sun} - m\,\underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(27)$$

Eq. (27) means that the force as measured on the earth is that due to the sun and the centrifugal force. The force due to the sun is directed towards the sun, and the centrifugal force directed oppositely.

Eq. (26) means that the force as measured on the sun is that due to the earth and the centripetal force. The force due to the earth is directed away from the sun and the centripetal force towards the sun.

The description of static and rotating frame is interchangeable. The earth can be viewed as rotating with respect to the static sun. However the sun can be viewed as rotating with respect to a static earth.

The important point is that the centripetal and centrifugal forces are real forces, not "pseudo forces". They can be analysed in terms of Cartan geometry as in UFT 55
