237(5): The Cotes and Poinsot Spirals and Frenet Analysis of the Kinematics of Orbits

It has been shown that the Coriolis acceleration vanishes in all orbits, so the force law of all orbits is:

$$\underline{F}(r) = m\left(\frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega}\times(\underline{\omega}\times\underline{r})\right) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right)\underline{e}_r \quad -(1)$$

where

$$\underline{L} = \underline{r}\times\underline{p}, \quad -(2)$$

$$L = |\underline{L}| = mr^2\omega. \quad -(3)$$

The magnitude of the spin connection is:

$$\omega = |\omega \underline{k}| = \frac{d\theta}{dt}. \quad -(4)$$

So the force depends entirely on the spin connection and the rotation of plane polar axes.

The two components of the force are:

1) $$\underline{F}_1 = \frac{d^2 r}{dt^2}\underline{e}_r = -\frac{L^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_r \quad -(5)$$

and:

2) $$\underline{F}_2 = m\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) = -\frac{L^2}{mr^3}\underline{e}_r \quad -(6)$$

Note carefully that the inverse square law of

2) Hooke and Newton is the sum of $\underline{F}_1$ and $\underline{F}_2$. The second type of force law (6) does not depend on the $r(\theta)$ function, it is the same for all orbits in a plane and is known as the centrifugal force. The latter is due to the spin connection ω and to the rotation of axes. The first type of force $\underline{F}_1$ is the inertial force and does not depend on the rotation of axes.

The centrifugal force does not exist in a static frame in which the axes are constant. Therefore the Hooke / Newton inverse square law applied to the attraction between m and M is:

$$\underline{F} = \underline{F}_1 = -\frac{mMG}{r^2}\underline{e}_r \quad -(7)$$

However, for an elliptical orbit:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad -(8)$$

and

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\theta \quad -(9)$$

so

$$\underline{F}_1 = \frac{L^2}{mr^2}\frac{\epsilon}{\alpha}\cos\theta\,\underline{e}_r \quad -(10)$$

$$= \frac{L^2}{mr^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right)\underline{e}_r$$

3)

Therefore when the axes start moving the force F_1 becomes different. The total force for an orbit of type (8) is:

$$\underline{F} = \underline{F_1} + \underline{F_2}$$

$$= \frac{L^2}{mr^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right)\underline{e}_r - \frac{L^2}{mr^3}\underline{e}_r$$

$$= -\frac{L^2}{mr^2\alpha}\underline{e}_r. \quad -(11)$$

$$= m\frac{d^2r}{dt^2}\underline{e}_r + m\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

The origin of the appellation "universal gravitation" is that eq. (11) has a similar structure to eq. (7). However eq. (7) contains no centrifugal acceleration. In order for eq. (11) to become (7):

$$\alpha = \frac{L^2}{m^2MG}. \quad -(12)$$

Universal gravitation exists if and only if α is assumed to be eq. (12). This is true only for an elliptical orbit. The observed orbits of astronomy are not elliptical.

For example the Cotes spirals are:

$$\frac{1}{r} = A\cos(k\theta + \epsilon), \quad -(13)$$

$$\frac{1}{r} = A\cosh(k\theta + \epsilon), \quad -(14)$$

$$\frac{1}{r} = A\theta + \epsilon \quad -(15)$$

and the Poinsot spirals are:

$$\frac{1}{r} = \frac{1}{a}\cosh(n\theta), \quad -(16)$$

$$\frac{1}{r} = \frac{1}{a}\sinh(n\theta). \quad -(17)$$

From eq. (13):

$$\underline{F}(r) = -\frac{L^2}{mr^3}(1 - k^2)\,\underline{e}_r \quad -(18)$$

so:

$$\underline{F}_1 = \frac{k^2L^2}{mr^3}, \quad \underline{F}_2 = -\frac{L^2}{mr^3} \quad -(19)$$

and

$$\underline{F} = \left(\frac{k^2L^2}{mr^3} - \frac{L^2}{mr^3}\right)\underline{e}_r \quad -(20)$$

$$= \frac{k^2L^2}{mr^3}\,\underline{e}_r + m\,\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

The "universal" law of gravitation no longer holds, but the centrifugal force is the same.

For the Poinsot spiral (16):

5)

$$\underline{F}(r) = -\frac{L^2}{mr^3}\left(1 + k^2\right)\underline{e}_r \quad -(21)$$

$$\underline{F_1} = -\frac{k^2 L^2}{mr^3}\,\underline{e}_r \quad -(22)$$

$$\underline{F_2} = m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\frac{L^2}{mr^3}\,\underline{e}_r \quad -(23)$$

The Cotes spiral (14) is the Poinsot spiral (16) if:

$$A = \frac{1}{a}, \quad k = n, \quad \epsilon = 0, \quad -(24)$$

and the Cotes spiral (13) written in the form:

$$f(\theta) = \frac{\epsilon}{\alpha}\cos(x\theta) \quad -(25)$$

becomes the precessing ellipse if:

$$\frac{1}{r} = \frac{1}{\alpha} + f(\theta). \quad -(26)$$

So the precessing ellipse is the Poinsot spiral (25) added to $1/\alpha$.

For the precessing ellipse:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -x^2\frac{\epsilon}{\alpha}\cos(x\theta) \quad -(27)$$

so from eq. (5):

$$\underline{F_1} = \frac{L^2 x^2 \epsilon}{m \, d \, r^2} \cos(x\theta) \, \underline{e}_r$$

$$= \frac{L^2 x^2}{m r^2 d} \left(\frac{d}{r} - 1 \right) \underline{e}_r$$

$$= \frac{L^2 x^2}{m r^2} \left(\frac{1}{r} - \frac{1}{d} \right) \underline{e}_r \quad -(28)$$

so the complete force law is:

$$\underline{F} = \left(x^2 \frac{L^2}{m r^2} \left(\frac{1}{r} - \frac{1}{d} \right) - \frac{L^2}{m r^3} \right) \underline{e}_r$$

$$= \left((x^2 - 1) \frac{L^2}{m r^3} - \frac{x^2 L^2}{m r^2 d} \right) \underline{e}_r \quad -(29)$$

which is the sum of inverse square and cube terms.

Each orbit has its different force law and there is no universal gravitation.

However, the centrifugal force:

$$\underline{F} = m \, \underline{\omega} \times (\underline{\omega} \times \underline{r}) = - \frac{L^2}{m r^3} \underline{e}_r \quad -(30)$$

is the same for all planar orbits. The force $\underline{F}$ is directed towards the origin in the observer frame of reference. In the solar system the origin is at M, the mass of

7) the sun. The force is directed inward because it is the force needed to counterbalance the outward directed centrifugal force $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$ of the planet:

$$\underline{F}\,(\text{outward}) = -m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \frac{L^2}{mr^3}\,\underline{e}_r \quad -(31)$$

The force $\underline{F}$ (outward) is the everyday force experienced in the hammer in any rotational motion. For example in athletics an inward force:

$$\underline{F}\,(\text{inward}) = m\underline{\omega} \times (\omega \times \underline{r}) = -\frac{L^2}{mr^3}\,\underline{e}_r \quad -(32)$$

is exerted on the hammer. When this is released, it accelerates away from the athlete. A planet in any planar orbit always tries to move in a straight line tangential to the orbit, but is prevented from doing so by the inward force (32) exerted by the sun.

All planar orbits can be analyzed by the Frenet formulae:

$$\underset{\sim}{T} = \frac{d\underline{r}}{dt} \quad -(33)$$

$$\frac{d\underset{\sim}{T}}{dt} = \frac{d^2\underline{r}}{dt^2} = \frac{1}{\rho}\,\underline{N} = \kappa\underline{N} \quad -(34)$$

$$\frac{d\underline{B}}{dt} = -\tau\underline{N} \quad -(35)$$

$$\frac{d\underline{N}}{dt} = -\frac{1}{\rho}\underset{\sim}{T} + \tau\underline{B} \quad -(36)$$

Here $\underset{\sim}{T}$ is the tangent vector, $\underline{N}$ is the normal vector, κ is the Frenet curvature and τ the Frenet torsion. Therefore the Frenet curvature is:

$$\kappa = |\dot{\underline{r}} \times \ddot{\underline{r}}| / |\dot{\underline{r}}|^3 \quad -(37)$$

and the Frenet torsion is:

$$\tau = \frac{\dot{\underline{r}} \cdot \ddot{\underline{r}} \times \dddot{\underline{r}}}{|\dot{\underline{r}} \times \ddot{\underline{r}}|^2} \quad -(38)$$

the binormal unit vector of Frenet is:

$$\underline{B} = \underline{T} \times \underline{N}. \quad -(39)$$

For the hyperbolic spiral orbit:

$$\underline{r} = -\frac{r_0}{\theta} \quad -(40)$$

the velocity is:

$$\underline{v} = \underline{T} = \frac{L}{m}\left(\frac{1}{r_0}\underline{e}_r + \frac{1}{r}\underline{e}_\theta\right) -(41)$$

and the acceleration is:

$$\underline{a} = \frac{1}{\rho}\underline{N} = -\frac{L^2}{m^2 r^3}\underline{e}_r \quad -(42)$$

9) Therefore:

$$\underline{v} \times \underline{a} = -\frac{L}{mr}\underline{e}_\theta \times \left(\frac{L^2}{m^2 r^3}\right)\underline{e}_r$$

$$= -\frac{L^3}{mr^4}\underline{e}_\theta \times \underline{e}_r$$

$$= \frac{1}{r}\left(\frac{L}{mr}\right)^3 \underline{k} \quad -(43)$$

Using

$$L = mr^2\omega \quad -(44)$$

then

$$\boxed{\underline{v} \times \underline{a} = r^2\omega^3\underline{k}} \quad -(45)$$

The Frenet curvature of the hyperbolic spiral is:

$$\boxed{\kappa = \frac{|\underline{v} \times \underline{a}|}{|\underline{v}|^3} = \frac{1}{r}\left(\frac{\omega r}{v}\right)^3} \quad -(46)$$

The binormal is:

$$\underline{B} = \underline{T} \times \underline{N} = \rho\,\underline{v} \times \underline{a} = \frac{\rho}{r}\left(\frac{\omega r}{v}\right)^3\underline{k} \quad -(47)$$

The normal is:

$$\underline{N} = \rho\,\underline{a} = -\rho\frac{L^2}{m^2r^3}\underline{e}_r \quad -(48)$$

By definition:

$$\underline{T} = \frac{dr}{dt} = \frac{dX}{dt}\underline{i} + \frac{dY}{dt}\underline{j} \quad -(49)$$

10)

and $$\frac{d\underline{T}}{dt} = \kappa \underline{N} \quad - (50)$$

so $$N_x = \frac{1}{\kappa}\frac{dT_x}{dt} \quad - (51)$$

$$N_y = \frac{1}{\kappa}\frac{dT_y}{dt} \quad - (52)$$

$$N_z = 0 \quad - (53)$$

Therefore:

$$\underline{B} = \underline{T} \times \underline{N} = (T_y N_z - T_z N_y)\underline{i} + (T_z N_x - T_x N_z)\underline{j} + (T_x N_y - T_y N_x)\underline{k} \quad - (54)$$

$$= (T_x N_y - T_y N_x)\underline{k}$$

Finally: $$\frac{d\underline{B}}{dt} = -\tau \underline{N} \quad - (55)$$

so $$\left(0, 0, \frac{dB_z}{dt}\right) = -\tau\left(\frac{dN_x}{dt}, \frac{dN_y}{dt}, 0\right)$$

so for all planar orbits the Frenet torsion vanishes.