237(9): Euler Bernoulli Resonance Equations in Plane Polar Coordinates.

Consider the acceleration in plane polar coordinates:

$$\frac{d^2 \underline{r}}{dt^2} = \frac{d^2 r}{dt^2} \underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(1)$$

where

$$\underline{\omega} = \frac{d\theta}{dt} \underline{k}. \quad -(2)$$

In order to simplify the problem consider a dynamical situation where the Coriolis force is zero, so:

$$\frac{d^2 \underline{r}}{dt^2} = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}). \quad -(3)$$

Such a situation occurs in all planar orbits. The acceleration in a frame of reference that is not rotating is:

$$\frac{d^2 r}{dt^2} \underline{e}_r = \frac{d^2 \underline{r}}{dt^2} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(4)$$

$$= \frac{d^2 \underline{r}}{dt^2} + \omega^2 r \underline{e}_r$$

$$= \frac{d^2 \underline{r}}{dt^2} + \omega^2 \underline{r}$$

2) The left hand side of eq. (4) is the acceleration in the Cartesian frame of reference. So:

$$\boxed{\left(\frac{d^2\underline{r}}{dt^2}\right)_{\text{Cartesian}} = \left(\frac{d^2\underline{r}}{dt^2} + \omega^2\underline{r}\right)_{\text{plane polar}}} \quad -(5)$$

if the Coriolis force is neglected for simplicity.

Eq. (5) becomes an Euler Bernoulli resonance equation if:

$$\frac{d^2\underline{r}}{dt^2} + \omega^2\underline{r} = A\cos\omega_1 t\, \underline{e}_r \quad -(6)$$

in plane polar coordinates.

Consider orbital motion in a plane. Then:

$$\underset{\sim}{r} = X\underline{i} + Y\underline{j} \quad -(7)$$

$$= r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j}$$

and

$$\underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \quad -(8)$$

So equating components in eq. (6):

$$\frac{d^2}{dt^2}(r\cos\theta) + \omega^2 r\cos\theta = A\cos\omega_1 t\cos\theta \quad -(9)$$

3)
$$\frac{d^2}{dt^2}(r\sin\theta) + \omega^2 r\sin\theta = A\cos\omega_1 t\sin\theta \quad -(10)$$

i.e
$$\boxed{\begin{aligned}\frac{d^2X}{dt^2} + \omega^2 X &= A_x\cos\omega_1 t\\ \frac{d^2Y}{dt^2} + \omega^2 Y &= A_y\cos\omega_1 t\end{aligned}}\quad \begin{aligned}-(11)\\-(12)\end{aligned}$$

where
$$A_x = A\cos\theta \quad -(13)$$
$$A_y = A\sin\theta \quad -(14)$$

Eqs. (11) and (12) are Euler Bernoulli resonance equations, QED.

The solutions of eqs. (11) and (12) are:
$$x(t) = \frac{A_x\cos\omega_1 t}{\omega^2-\omega_1^2} \quad -(15)$$

and
$$y(t) = \frac{A_y\cos\omega_1 t}{\omega^2-\omega_1^2} \quad -(16)$$

The resonance condition is
$$\boxed{\omega = \omega_1}, \quad -(17)$$

At which point:
$$\boxed{r \rightarrow \infty} \quad -(18)$$

4)

From a mathematical point of view the resonance is caused by changing the coordinate system from Cartesian to plane polar. There is no resonance in the Cartesian system, but there is resonance in the plane polar system.

Similarly, the Coriolis and centripetal acceleration exist only in the plane polar system, they do not exist in the Cartesian system.

Therefore new physics emerges from using the plane polar system. It should be possible to devise an experiment to observe the resonance in a rotating particle subjected to a driving force. In atomic theory, resonance occurs when the driving force is electromagnetic radiation, and the rotating object is an electron. In general, the acceleration in plane polar coordinates is:

$$\underline{a} = \frac{D\underline{v}}{dt} = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} \quad -(19)$$

where D/dt is the covariant derivative and $\underline{\omega}$ the spin connection.