237(4): Logarithmic Spiral Kinematics

In this note it is shown that the kinematics of a logarithmic spiral are not consistent with a constant velocity as r becomes infinite. The logarithmic spiral trajectory is defined by:

$$r = r(0)\exp(\alpha\theta) \quad -(1)$$

where $r(0)$ and α are constants. The force of attraction for the trajectory is given by:

$$\underline{F}(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \quad -(2)$$

From eqs. (1) and (2):

$$\underset{\sim}{F}(r) = -(1+\alpha^2)\frac{L^2}{mr^3}\underline{e}_r \quad -(3)$$

Therefore both a logarithmic and hyperbolic spiral are given by an inverse cubed force law.

However the limiting velocity is given by:

$$\underline{v} = \left(\frac{L}{m}\right)\left(\frac{1}{r}\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right), \quad -(4)$$

$$v^2 = \left(\frac{L}{m}\right)^2\left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right) \quad -(5)$$

For the logarithmic spiral:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\alpha}{r} \quad -(6)$$

2)

so: $$v^2 = \left(\frac{L}{m}\right)^2 \left(\frac{1+\alpha^2}{r^2}\right) \quad -(7)$$

and $$\boxed{v \xrightarrow[r \to \infty]{} 0} \quad -(8)$$

This is contrary to the experimental observation.

For the hyperbolic spiral:

$$r = -\frac{r_0}{\theta} \quad -(9)$$

then $$v^2 = \left(\frac{L}{m}\right)^2 \left(\frac{1}{r^2} + \frac{1}{r_0^2}\right) \quad -(10)$$

and $$v \xrightarrow[r \to \infty]{} \frac{L}{m r_0} = v_\infty \quad -(11)$$

which is consistent with the experimental data.

For the logarithmic spiral:

$$\frac{d\theta}{dt} = \omega = \frac{L}{mr^2} = \frac{L}{m\alpha^2 r^2(0)} e^{-2\alpha\theta} \quad -(12)$$

so $$t = \int dt = \int \frac{L}{m\alpha^2 r^2(0)} e^{-2\alpha\theta} d\theta$$

$$= \frac{m r^2(0)}{2\alpha L} e^{2\alpha\theta} \quad -(13)$$

3) i.e
$$t = \frac{mr^2}{2\alpha L} \quad -(14)$$

so
$$\boxed{r(t) = \left(\frac{2\alpha L}{m} t\right)^{1/2}} \quad -(15)$$

The radial coordinate increases with time. This means that the orbiting star travels along a logarithmic spiral from near the centre outwards.

For the hyperbolic spiral:
$$\frac{d\theta}{dt} = \frac{L}{mr^2} = \frac{L}{mr_0^2}\theta^2 \quad -(16)$$
$$dt = \frac{mr_0^2}{L}\frac{d\theta}{\theta^2} \quad -(17)$$
and
$$t = \frac{mr_0^2}{L}\int \frac{d\theta}{\theta^2} = -\frac{mr_0^2}{L}\frac{1}{\theta}$$
$$\boxed{t = \frac{mr_0 r}{L}} \quad -(18)$$

the radial coordinate r increases with time, and the star moves outwards from near the centre on a hyperbolic spiral.