257(6): Frenet Analysis of Planar Orbits

In the Frenet analysis the curve $\underset{\sim}{r}$ is parameterized as follows:

$$\underline{r} = \underline{r}(s) \quad -(1)$$

and the tangent and normal vectors are defined as:

$$\underset{\sim}{T} = \frac{d\underline{r}}{ds}, \quad \underline{N} = \rho \frac{d\underset{\sim}{T}}{ds} \quad -(2)$$

where ρ is the radius of curvature. In the plane polar system:

$$\underset{\sim}{r} = r\cos\theta\, \underline{i} + r\sin\theta\, \underline{j}. \quad -(3)$$

Now define:

$$s = r \quad -(4)$$

so

$$\underset{\sim}{T} = \frac{d\underline{r}}{dr} = \cos\theta\, \underline{i} + \sin\theta\, \underline{j} = \underline{e}_r. \quad -(5)$$

The normal unit vector is defined by:

$$\underline{N} = \rho \frac{d\underset{\sim}{T}}{dr} = \rho \frac{d\underline{T}}{d\theta}\frac{d\theta}{dr} = \rho \frac{d\theta}{dr}\underline{e}_\theta \quad -(6)$$

Since $\underline{N}$ and $\underline{e}_\theta$ are unit vectors:

$$\boxed{\rho = \frac{dr}{d\theta}} \quad -(7)$$

The curvature ρ of the Frenet system can be defined for any planar orbit by $dr/d\theta$. Therefore:

$$\rho = \frac{dr}{d\theta} = \frac{ds}{d\theta} \quad -(8)$$

and:

2)

$$\underset{\sim}{T} = \underline{e}_r, \quad \underline{N} = \underline{e}_\theta. \quad -(9)$$

The velocity is defined by:

$$\underline{v} = v\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta$$

$$= v\underset{\sim}{T} + r\omega\,\underline{N} \quad -(10)$$

$$= v\underset{\sim}{T} + r\frac{d\theta}{dr}\frac{dr}{dt}N.$$

$$\boxed{\underline{v} = v\underline{T} + \frac{vr}{\rho}\underline{N}} \quad -(11)$$

The binormal unit vector is defined by:

$$\underline{B} = \underline{T} \times \underline{N} = \underline{e}_r \times \underline{e}_\theta = \underline{k} \quad -(12)$$

The third Frenet formula is:

$$\frac{d\underline{N}}{ds} = \tau\underline{B} - \frac{1}{\rho}\underset{\sim}{T} \quad -(13)$$

where τ is the Frenet torsion. So:

$$\frac{d\underline{e}_\theta}{dr} = \tau\underline{k} - \frac{1}{\rho}\underline{e}_r \quad -(14)$$

$$= \frac{d\underline{e}_\theta}{dt}\frac{dt}{dr}$$

$$= -\frac{\omega}{v}\underline{e}_r$$

It follows that:

3)

$$\boxed{\rho = \frac{v}{\omega}, \quad \tau = 0} \quad - (15)$$

for all planar orbits. For a circular orbit:

$$\rho = r \quad - (16)$$

otherwise the Frenet curvature is not the radius r.

The acceleration is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad - (17)$$

The tangential and normal components of velocity are defined by:

$$\underline{v} = v_T \underset{\sim}{T} + v_N \underline{N} \quad - (18)$$

i.e.

$$v_T = \frac{dr}{dt}, \quad v_N = r\omega. \quad - (19)$$

The tangential acceleration is:

$$\underline{a}_T = \ddot{r}\,\underline{e}_r + \dot{r}\dot{\theta}\,\underline{e}_\theta \quad - (20)$$

$$= \frac{dv}{dt}\underline{T} + \frac{v^2}{\rho}\underline{N}.$$

The normal acceleration is:

$$\underline{a}_N = (\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_\theta - r\dot{\theta}^2\,\underline{e}_r \quad - (21)$$

$$= \frac{v^2}{\rho}\underline{N} - r\omega^2\,\underline{e}_r + r\ddot{\theta}\,\underline{e}_\theta$$

Note that:

$$\omega^2 = \left(\frac{d\theta}{dr}\right)^2 \left(\frac{dr}{dt}\right)^2 = \left(\frac{v}{\rho}\right)^2 \quad -(22)$$

so

$$\boxed{v = \omega\rho} \quad -(23)$$

for any orbit in a plane.

Therefore:

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{dt}\left(\frac{d\theta}{dr}\frac{dr}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{v}{\rho}\right)$$

$$= \frac{1}{\rho}\frac{dv}{dt} + v\frac{d}{dt}\left(\frac{1}{\rho}\right) \quad -(24)$$

$$= \frac{1}{\rho}\frac{dv}{dt} - \frac{v}{\rho^2}\frac{d\rho}{dt}$$

Therefore:

$$\boxed{\underline{a} = \left(\frac{dv}{dt} - r\left(\frac{v}{\rho}\right)^2\right)\underset{\sim}{T} + \left(\frac{r}{\rho}\frac{dv}{dt} - \frac{vr}{\rho^2}\frac{d\rho}{dt} + \frac{2v^2}{\rho}\right)\underline{N}} \quad -(25)$$

The inertial acceleration is:

$$\underline{a}_{in} = \frac{dv}{dt}\underset{\sim}{T} \quad -(26)$$

5) The centrifugal acceleration is:

$$\underline{a}_{cent} = -r\left(\frac{v}{\rho}\right)^2 \underset{\sim}{T} \quad -(27)$$

The Coriolis acceleration is:

$$\underline{a}_{Cor} = \left(\frac{2v^2}{\rho} + \frac{r}{\rho}\frac{dv}{dt} + vr\frac{d}{dt}\left(\frac{1}{\rho}\right)\right)\underline{N}$$

$$-(28)$$

The Coriolis acceleration vanishes in all planar orbits.