235(9): Spin Tensor Elements of the Plane Polar Coordinates

The plane polar coordinates are defined by:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta, \quad -(1)$$

$$\underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \quad -(2)$$

so

$$\frac{d\underline{e}_r}{dr} = \left(\frac{d\theta}{dr}\right)\underline{e}_\theta \quad -(3)$$

$$\frac{d\underline{e}_r}{d\theta} = \underline{e}_\theta \,. \quad -(4)$$

The infinitesimal line element is defined by:

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = g_{11}\,dx^1 dx^1 + g_{22}\,dx^2 dx^2$$

$$= dr^2 + r^2 d\theta^2 \quad -(5)$$

so

$$x^1 = r, \quad x^2 = r\theta, \quad -(6)$$

$$\partial_1 = \frac{\partial}{\partial r}, \quad \partial_2 = \frac{\partial}{\partial(r\theta)} \,. \quad -(7)$$

Note that if:

$$y = r\theta \quad -(8)$$

then

$$\frac{df}{dy} = \frac{df}{d\theta}\frac{d\theta}{dr} \quad -(9)$$

so

$$\frac{d\underline{e}_r}{d(r\theta)} = \frac{d\theta}{dr}\underline{e}_\theta \quad -(10)$$

2)

Eq. (3) may be written as:

$$\frac{d e^{(1)}}{d x^{1}} = \omega^{(1)}{}_{1(2)} e^{(2)} \quad -(11)$$

so

$$\omega^{(1)}{}_{1(2)} = \frac{d\theta}{dr} . \quad -(12)$$

Eq. (10) may be written as:

$$\frac{d e^{(1)}}{d x^{2}} = \omega^{(1)}{}_{2(2)} e^{(2)} \quad -(13)$$

so

$$\omega^{(1)}{}_{2(2)} = \frac{d\theta}{dr} . \quad -(14)$$

Therefore the spin connections are:

$$\boxed{\omega^{(1)}{}_{1(2)} = \omega^{(1)}{}_{2(2)} = \frac{d\theta}{dr}} \quad -(15)$$

From note 235(7) the tetrads are:

$$q^{(1)}{}_{1} = \cos\theta , \quad q^{(1)}{}_{2} = \sin\theta , \quad -(16)$$

$$q^{(2)}{}_{1} = -\sin\theta , \quad q^{(2)}{}_{2} = \cos\theta .$$

The Cartan torsion is:

$$T^{(a)}{}_{\mu\nu} = -T^{(a)}{}_{\nu\mu} = \partial_\mu q^{(a)}{}_{\nu} - \partial_\nu q^{(a)}{}_{\mu} + \omega_{\mu}{}^{(a)}{}_{(b)} q^{(b)}{}_{\nu} - \omega_{\nu}{}^{(a)}{}_{(b)} q^{(b)}{}_{\mu} \quad -(17)$$

Therefore:

$$T^{(1)}_{12} = \frac{d\sin\theta}{dr} - \frac{d\cos\theta}{d(r\theta)} + \omega^{(1)}_{1(1)} q^{(1)}_{2} + \omega^{(1)}_{1(2)} q^{(2)}_{2} - \omega^{(1)}_{2(1)} q^{(1)}_{1} - \omega^{(1)}_{2(2)} q^{(2)}_{1} \quad -(18)$$

in which:

$$\omega^{(1)}_{1(1)} = \omega^{(1)}_{2(1)} = 0. \quad -(19)$$

So

$$T^{(1)}_{12} = \frac{d\theta}{dr}\cos\theta + \frac{d\theta}{dr}\sin\theta + \frac{d\theta}{dr}\cos\theta + \frac{d\theta}{dr}\sin\theta$$

$$= 2\frac{d\theta}{dr}\left(\cos\theta + \sin\theta\right) \quad -(20)$$

Note carefully that antisymmetry is obeyed:

$$\partial_\mu q^{(a)}_{\nu} + \omega^{(a)}_{\mu(b)} q^{(b)}_{\nu} = -\left(\partial_\nu q^{(a)}_{\mu} + \omega^{(a)}_{\nu(b)} q^{(b)}_{\mu}\right) \quad -(21)$$

Similarly:

$$T^{(1)}_{21} = \partial_2 q^{(1)}_{1} - \partial_1 q^{(1)}_{2} + \omega^{(1)}_{2(2)} q^{(2)}_{1} - \omega^{(1)}_{1(2)} q^{(2)}_{2}$$

$$= \frac{d}{d(r\theta)}\cos\theta - \frac{d\sin\theta}{dr} - \frac{d\theta}{dr}\sin\theta - \frac{d\theta}{dr}\cos\theta$$

$$= -\frac{d\theta}{dr}\left(\sin\theta + \cos\theta + \sin\theta + \cos\theta\right) \quad -(22)$$

$$= -2\frac{d\theta}{dr}\left(\cos\theta + \sin\theta\right)$$

$$= -T^{(1)}_{12}$$

QED

4) From note 235(7) the orbital torsion elements are:

$$\begin{bmatrix} T^{(1)}_{01} & T^{(1)}_{02} \\ T^{(2)}_{01} & T^{(2)}_{02} \end{bmatrix} = 2\omega \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad -(23)$$

In order to make the orbital and spin torsion components dimensionally consistent and correctly relativistic:

$$\begin{bmatrix} T^{(1)}_{01} & T^{(1)}_{02} \\ T^{(2)}_{01} & T^{(2)}_{02} \end{bmatrix} = \frac{2\omega}{c} \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad -(24)$$

where $c$ is the vacuum speed of light.

An orbit of any kind is defined by $dr/d\theta$ if it is a planar orbit. For an elliptical orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(25)$$

so

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(26)$$

and

$$\frac{d\theta}{dr} = \frac{\alpha}{\epsilon r^2 \sin\theta} \quad -(27)$$

so

$$\boxed{T^{(1)}_{12} = -T^{(1)}_{21} = 2\,\frac{\alpha}{\epsilon r^2}\left(1 + \cot\theta\right)} \quad -(28)$$