235(6): Spin Connection in the Fundamental Rotation of Axes.

In the plane polar coordinate system:

$$\underline{e}_r = \underline{i}\cos\theta(t) + \underline{j}\sin\theta(t) \quad -(1)$$

$$\underline{e}_\theta = -\sin\theta(t)\,\underline{i} + \cos\theta(t)\,\underline{j} \quad -(2)$$

Therefore $\underline{e}_r$ and $\underline{e}_\theta$ depend on time, and the unit vectors rotate. The spin connection defines this rotation. In tensor notation the basis vectors are:

$$e_r = e^{(1)}, \quad e_\theta = e^{(2)} \quad -(3)$$

The covariant derivative is defined by:

$$D_\mu e^{(a)} = \partial_\mu e^{(a)} + \omega^{(a)}{}_{\mu(b)} e^{(b)} \quad -(4)$$

for example

$$\frac{De^{(a)}}{dt} = \left(\frac{de^{(a)}}{dt}\right)_{static} + \omega^{(a)}{}_{1(b)} e^{(b)} \quad -(5)$$

However:

$$\left(\frac{de^{(a)}}{dt}\right)_{static} = 0 \quad -(6)$$

because for static coordinates θ is not time dependent. So for the coordinates:

$$\frac{De^{(a)}}{dt} = \omega^{(a)}{}_{1(b)} e^{(b)} \quad -(7)$$

2)

In vector notation:

$$\boxed{\frac{D\underline{e}_r}{dt} = \omega^{(1)}_{1(2)}\,\underline{e}_\theta} \quad -(8)$$

This result is always denoted by:

$$\frac{d\underline{e}_r}{dt} = \omega\,\underline{e}_\theta \quad -(9)$$

but rigorously, it should be: $\quad -(10)$

$$\boxed{\frac{D\underline{e}_r(t)}{dt} = \left(\frac{d\underline{e}_r}{dt}\right)_{static} + \omega\,\underline{e}_\theta(t)}$$

It follows that:

$$\boxed{\omega^{(1)}_{1(2)} = \omega} \quad -(11)$$

a result of basic importance. The fundamental equation (10) can be expressed as:

$$\frac{D\underline{e}_r(t)}{dt} = \frac{d\underline{e}_r(\text{static})}{dt} + \underline{\omega}\times\underline{e}_r$$

$-(12)$

if:

$$\boldsymbol{\omega} = \omega \mathbf{k} \quad -(13)$$

and

$$\mathbf{k} = \mathbf{e}_r \times \mathbf{e}_\theta \quad -(14)$$

$$\mathbf{e}_\theta = \mathbf{k} \times \mathbf{e}_r \quad -(15)$$

The spin connection vector $\boldsymbol{\omega}$ is fundamental to all dynamics.

Similarly:

$$\frac{D\mathbf{r}}{dt} = \left(\frac{d\mathbf{r}}{dt}\right)_{static} + \boldsymbol{\omega} \times \mathbf{r} \quad -(16)$$

where

$$\mathbf{r} = r\,\mathbf{e}_r, \quad -(17)$$

i.e.

$$\frac{D}{dt}(r\,\mathbf{e}_r) = \left(\frac{d}{dt}(r\,\mathbf{e}_r)\right)_{static} + r\,\boldsymbol{\omega} \times \mathbf{e}_r$$

$$= \frac{dr}{dt}\mathbf{e}_r + \boldsymbol{\omega} \times \mathbf{r} \quad -(18).$$

In the theory of angular momentum $\mathbf{L}$:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad -(19)$$

$$= m\,\mathbf{r} \times \mathbf{v}.$$

Here

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad -(20)$$

4) The torque is therefore:

$$\underline{Tq} = \frac{d\underline{L}}{dt} = m\left(\frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times \underline{a}\right) \quad -(21)$$

where

$$a = \frac{d\underline{v}}{dt}. \quad -(22)$$

So

$$\underline{Tq} = \frac{d\underline{L}}{dt} = m\,\underline{r} \times \underline{a} \quad -(23)$$

The acceleration is:

$$\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + 2\underline{v}_s \times \underline{\omega} \quad -(24)$$

$$= \left(\frac{d^2 r}{dt^2} - \omega^2 r\right)\underline{e}_r + \left(r\frac{d\omega}{dt} + 2\omega\frac{dr}{dt}\right)\underline{e}_\theta$$

In a static or inertial frame:

$$\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r \quad -(25)$$

so the torque is due entirely to:

$$\underline{Tq} = \frac{d\underline{L}}{dt} = m\,\underline{r} \times \left(r\frac{d\omega}{dt} + 2\omega\frac{dr}{dt}\right)\underline{e}_\theta \quad -(26)$$

5) i.e.

$$\underline{T_{q_r}} = \frac{d\underline{L}}{dt} = m\left(r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega \right)\underline{k} \quad -(27)$$

$$= 2(\underline{v_s} \times \underline{\omega}) \times \underline{r} + \underline{r} \times \left(\frac{d\underline{\omega}}{dt} \times \underline{r} \right)$$

for motion in a plane.

For any orbit:

$$\underline{T_{q_r}} = \underline{0} \quad -(28)$$

because $\underline{J}$ is a constant of motion.

In general the angular momentum can be expressed as:

$$\underline{L} = m\,\underline{r} \times (\underline{\omega} \times \underline{r}) = m\left(r^2\underline{\omega} - \underline{r}(\underline{r}\cdot\underline{\omega})\right) \quad -(29)$$

If: $\underline{\omega} = \omega\underline{k}$, $\underline{r} = r\underline{e}_r$ $-(30)$

then

$$\underline{L} = mr^2\underline{\omega} \quad -(31)$$

for motion in a plane. Otherwise $\underline{L} \not\parallel \underline{\omega}$.
For motion in a plane the angular momentum is the spin connection vector.