235(1): Torsion and Connection in Rotational Dynamics

In this note it is shown that the rotation in a plane is itself a connection - the rotation generator is related to the Cartan torsion directly. Therefore any axis it is a Cartan torsion. Consider the plane polar coordinate system for simplicity and clarity. The unit vectors of the system are:

$$\underline{e}_r = \cos\theta \, \underline{i} + \sin\theta \, \underline{j} \quad -(1)$$

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(2)$$

These are related by:

$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_\theta \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(3)$$

which is an example of the defining equation of the Cartan tetrad:

$$V^a = q^a_\mu V^\mu \quad -(4)$$

The tetrad components are:

$$q^{(1)}_1 = \cos\theta, \quad q^{(1)}_2 = \sin\theta, \quad -(5)$$

$$q^{(2)}_1 = -\sin\theta, \quad q^{(2)}_2 = \cos\theta$$

So the complete tetrad matrix is:

2)

$$V_\mu^a = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad -(6)$$

However, this is also the rotation matrix about Z:

$$\begin{bmatrix} V_x' \\ V_y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} \quad -(7)$$

It follows from eqs. (4) and (7) that:

$$\begin{bmatrix} V^{(1)} \\ V^{(2)} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \quad -(8)$$

where:

$$V^{(1)} = V_x', \quad V^{(2)} = V_y', \quad -(9).$$
$$V^1 = V_x, \quad V^2 = V_y.$$

Proof

$$\underline{V} = V_x \underline{i} + V_y \underline{j}$$
$$= V^{(1)} \underline{e}_r + V^{(2)} \underline{e}_\theta$$
$$= V^{(1)} (\cos\theta\, \underline{i} + \sin\theta\, \underline{j}) \quad -(10)$$
$$+ V^{(2)} (-\sin\theta\, \underline{i} + \cos\theta\, \underline{j})$$
$$= V^1 \underline{i} + V^2 \underline{j}$$

So:

$$V^1 = V^{(1)}\cos\theta - V^{(2)}\sin\theta \quad -(11)$$
$$V^2 = V^{(1)}\sin\theta + V^{(2)}\cos\theta \quad -(12)$$

Multiply eq. (11) by $\cos\theta$ and eq. (12) by $-\sin\theta$.

3) It follows that:

$$V^{(1)} = V^1 \cos\theta + V^2 \sin\theta \quad -(13)$$

$$V^{(2)} = -V^1 \sin\theta + V^2 \cos\theta \quad -(14)$$

which is eq. (8), QED.

It has been proven that rotation in a plane XY about Z defines a Cartan tetrad.

Define the metric in the Cartesian system as $g_{\mu\nu}$ and the metric in the plane polar system as η_{ab}. The two metrics are related by:

$$g_{\mu\nu} = q^a_\mu \, q^b_\nu \, \eta_{ab} \quad -(15)$$

by fundamental definition of the vielbein or tetrad as factorizing the metric. The metric is related to the infinitesimal line element by:

$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu, \quad -(16)$$

and

$$ds^2 = \eta_{ab} \, dx^a dx^b. \quad -(17)$$

From fundamentals ("Vector Analysis Problem Solver"):

$$ds^2 = dX^2 + dY^2 = dr^2 + r^2 d\theta^2 \quad -(18)$$

4) In this system:

$$ds^2 = g_{11}\, dx^1 dx^1 + g_{22}\, dx^2 dx^2 \quad -(19)$$

In Cartesian coordinates:

$$dx^1 = dX, \quad dx^2 = dY, \quad -(20)$$

$$g_{11} = g_{22} = 1$$

In plane polar coordinates:

$$dx^1 = dr, \quad dx^2 = r\, d\theta \quad -(21)$$

$$g_{11} = 1, \quad g_{22} = 1$$

$$-(22)$$

Eq. (15) means:

$$g_{11} = q^{(1)}_{\ 1} q^{(1)}_{\ 1} \eta_{(1)(1)} + q^{(2)}_{\ 1} q^{(2)}_{\ 1} \eta_{(2)(2)}$$

$$g_{22} = q^{(1)}_{\ 2} q^{(1)}_{\ 2} \eta_{(1)(1)} + q^{(2)}_{\ 2} q^{(2)}_{\ 2} \eta_{(2)(2)}$$

$$-(23)$$

From eqs. (5), (20) and (21), eqs. (22) and (23) are correct and self consistent. Both give:

$$\cos^2\theta + \sin^2\theta = 1 \quad -(24)$$

From <u>rotation generator theory</u>, if:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad -(25)$$

5) then the rotation generator is:

$$J_Z = \frac{1}{i} \frac{dR_Z}{d\theta}\bigg|_{\theta=0} \quad -(26)$$

$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad -(27)$$

In three dimensions:

$$J_Z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; J_X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \; J_Y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

and

$$[J_X, J_Y] = i J_Z \quad -(28)$$

et cyclicum

It follows that:

$$R_Z(\theta) = \exp(i J_Z \theta) \quad -(29)$$

Proof

$$e^{iJ_Z\theta} = 1 + i J_Z \theta - J_Z^2 \frac{\theta^2}{2!} + \ldots \quad -(30)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \theta \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\theta^2}{2!} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

QED.

Therefore the tetrad is:

$$\boxed{q^a_\mu = \exp\left(i J_Z \theta\right)} \quad -(31)$$

where both sides are matrices.

The general structure of eq. (26) is that of the derivative of a tetrad, because ~~the~~ the rotation matrix R_Z is a tetrad. For any vector field $\underline{V}$:

$$D_\nu q^a_\mu = 0 \quad -(32)$$

Eq. (32) means that:

$$\underline{V} = V^1 \underline{i} + V^2 \underline{j} \quad -(33)$$
$$= V^{(1)} \underline{e}_r + V^{(2)} \underline{e}_\theta$$

Eq. (32) means:

$$\partial_\nu q^a_\mu + \omega^a_{\ \nu b} q^b_\mu - \Gamma^\lambda_{\ \nu\mu} q^a_\lambda = 0, \quad -(34)$$

which can be expressed as:

$$\partial_\nu q^a_\mu = \Gamma^a_{\ \nu\mu} - \omega^a_{\ \nu\mu} \quad -(35)$$
$$:= \zeta^a_{\ \nu\mu}$$

Eq. (35) is a generalization of eq. (26).

7) Eq. (35) shows that the rotation generator is a special case of the connection $\Gamma^a_{\mu\nu}$. The Cartan torsion associated with this process is:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(36)$$

$$= \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu\nu} - \omega^a_{\nu\mu}$$

$$= \Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu}$$

Any orbit may therefore be thought of in terms of a connection and torsion.

There is a close connection between the orbit and the tetrad because in an orbit:

$$dr / d\theta \neq 0. \quad -(37)$$

It follows that:

$$\frac{d\cos\theta}{dr} = -\sin\theta \frac{d\theta}{dr} \quad -(38)$$

$$\frac{d\sin\theta}{dr} = \cos\theta \frac{d\theta}{dr} \quad -(39)$$

So from eq. (6):

$$\frac{d\,q_{V_\mu}^{\;a}}{dr} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \frac{d\theta}{dr} \quad -(40)$$

From eq. (35):

$$\xi^{a}_{1\mu} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \frac{d\theta}{dr} \quad -(41)$$

i.e.

$$\xi^{(1)}_{11} = -\sin\theta \frac{d\theta}{dr}, \quad \xi^{(1)}_{12} = \cos\theta \frac{d\theta}{dr},$$
$$\xi^{(2)}_{11} = -\cos\theta \frac{d\theta}{dr}, \quad \xi^{(2)}_{12} = -\sin\theta \frac{d\theta}{dr}. \quad -(42)$$

In general the orbit is proportional to the connection, which is a rotation generator proportional to angular momentum and related to spacetime torsion.

Elliptical Orbit

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta. \quad -(43)$$

Therefore:

$$\boxed{\begin{aligned} \xi^{(1)}_{11} = \xi^{(2)}_{12} &= -\frac{\alpha}{\epsilon r^2} \\ \xi^{(1)}_{12} = -\xi^{(2)}_{11} &= -\frac{\alpha}{\epsilon r^2}\,\mathrm{cotan}\,\theta. \end{aligned}} \quad -(44)$$