235(2): The Connection as Angular Velocity.

Consider the definition of linear velocity in the plane polar coordinate system:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d(r\underline{e}_r)}{dt} \quad -(1)$$

The coordinate system is rotating so:

$$\frac{d(r\underline{e}_r)}{dt} = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(2)$$

where

$$\frac{d\underline{e}_r}{dt} = \frac{d\theta}{dt}\underline{e}_\theta. \quad -(3)$$

Here

$$\omega = \frac{d\theta}{dt} \quad -(4)$$

is the magnitude of the angular velocity. In this system:

$$\underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \quad -(5)$$

$$\underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \quad -(6)$$

and

$$\underline{k} \times \underline{e}_r = \underline{e}_\theta. \quad -(7)$$

Therefore:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \omega r\,\underline{e}_\theta. \quad -(8)$$

Note carefully that the left hand side of this equation is a covariant derivative.

2) It can be written as:

$$\frac{D\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(9)$$

$$\boxed{\frac{D\underline{r}}{dt} = \left(\frac{d\underline{r}}{dt}\right)_{static} + \underline{\omega} \times \underline{r}} \quad -(10)$$

<u>Proof</u>

We have:

$$r\frac{d\underline{e}_r}{dt} = \underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta \quad -(11)$$

so $$\underline{\omega} \times r\underline{e}_r = \omega r \underline{e}_\theta \quad -(12)$$

and $$\underline{\omega} \times \underline{e}_r = \omega \underline{e}_\theta . \quad -(13)$$

From eq. (7): $$\underline{\omega} = \omega \underline{k} \quad -(14)$$

QED.

Eq. (10) is true for any vector $\underline{V}$ and is the fundamental and well known theorem of rotational dynamics:

$$\frac{D\underline{V}}{dt} = \left(\frac{d\underline{V}}{dt}\right)_{static} + \underline{\omega} \times \underline{V} \quad -(15)$$

) For ease of notation it can be written as:

$$\frac{D\underline{V}}{dt} = \frac{d\underline{V}}{dt} + \underline{\omega}\times\underline{V} \quad -(16)$$

Note carefully that eq. (16) is an example of the equation:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\ \mu b} V^b \quad -(17)$$

where $\omega^a_{\ \mu b}$ is the spin connection of Cartan.

For rotational motion the relevant element of the spin connection is the angular velocity.

Proof

Consider the complete vector field of the position vector, denoted by:

$$\underline{R} = r^{(1)}\,\underline{e}_{(1)} \quad -(18)$$

where

$$r^{(1)} = r, \quad \underline{e}_{(1)} = \underline{e}_r. \quad -(19)$$

Consider the covariant derivative of R with respect to time:

$$\frac{D\underline{R}}{dt} = \left(\frac{dr^{(1)}}{dt} + \omega^{(1)}_{\ 1(b)}r^{(b)}\right)\underline{e}_{(1)} + \left(\frac{dr^{(2)}}{dt} + \omega^{(2)}_{\ 1(b)}r^{(b)}\right)\underline{e}_{(2)} \quad -(20)$$

defined by Cartan geometry (Carroll Chapt 3).

4) In eq. (20):
$$r^{(2)} = 0, \quad -(21)$$

and
$$\frac{D\underline{R}}{dt} = \frac{dr}{dt}\,\underline{e}_{(1)} + r\frac{d\theta}{dt}\,\underline{e}_{(2)} \quad -(22)$$

where
$$\underline{e}_{(1)} = \underline{e}_r\,,\ \underline{e}_{(2)} = \underline{e}_\theta. \quad -(23)$$

It follows that:
$$\frac{D\underline{R}}{dt} = \frac{dr}{dt}\,\underline{e}_r + \omega^{(2)}{}_{1(2)}\,r\,\underline{e}_\theta \quad -(24)$$

i.e.
$$\boxed{\omega^{(2)}{}_{1(2)} = \omega} \quad -(25)$$

QED.

Therefore the velocity is:
$$\underline{v} = \frac{D}{dt}(r\,\underline{e}_r) = \dot{r}\,\underline{e}_r + \omega r\,\underline{e}_\theta \quad -(26)$$

where the angular velocity magnitude ω is a spin connection element $\omega^{(2)}{}_{1(2)}$. Eq. (26) is an example of Cartan geometry.

5) The symbol D emphasises that the linear velocity in the plane polar system is a <u>covariant derivative</u> because the axes are moving. Eq. (26) may be written as:

$$\boxed{\frac{D\underline{r}}{dt} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}} \quad -(27)$$

<u>where</u> $\underline{\omega}$ is <u>the vector spin connection</u>.

For any vector $\underline{V}$:

$$\frac{D\underline{V}}{dt} = \frac{d\underline{V}}{dt} + \underline{\omega} \times \underline{V} \quad -(28)$$

The acceleration is:

$$\underline{a} = \frac{D\underline{v}}{dt} = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} \quad -(29)$$

where

$$\underline{v} = \frac{D\underline{r}}{dt} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad -(30)$$

<u>The same spin connection occurs in eqs. (29) and (30).</u>

6) Therefore:

$$\underline{a} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}\right) + \underline{\omega} \times \left(\frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}\right) \quad -(31)$$

If $\underline{\omega}$ is a constant:

$$\frac{d\underline{\omega}}{dt} = \underline{0}, \quad -(32)$$

then:

$$\boxed{\underline{a} = \frac{d^2\underline{r}}{dt^2} + 2\,\underline{\omega} \times \frac{d\underline{r}}{dt} + \underline{\omega} \times (\underline{\omega} \times \underline{r})} \quad -(33)$$

Here

$$\underline{a}_{\text{Coriolis}} = 2\,\underline{\omega} \times \frac{d\underline{r}}{dt} \quad -(34)$$

$$\underline{a}_{\text{centrifugal}} = \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(35)$$

Conclusion

The Coriolis and centrifugal accelerations are due to the spin connection $\underline{\omega}$ of Cartan.