: The Acceleration as a Covariant Derivative.

This is a very important result of Carta geometry, which reveals very clearly the reason why Newtonian dynamics is incomplete and cannot describe orbits. The versa is that the convection is missing from Newtonian dynamics.

Consider the position vector in plane polar coordinates:

$$\underline{r} = r \, \underline{e}_r \quad - (1)$$

(see "Vector Analysis Problem Solver" and Maria and Thornta for the basics).

The linear velocity is:

$$\underline{v} = \frac{dr}{dt} = \frac{d}{dt}(r\underline{e}_r) = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad - (2)$$

where

$$\frac{d\underline{e}_r}{dt} = \frac{d\theta}{dt}\underline{e}_\theta = \omega\underline{e}_\theta \quad - (3)$$

therefore

$$\boxed{\underline{v} = \underline{v}_N + \omega r \underline{e}_\theta} \quad - (4)$$

The Newtonian velocity is:

$$\underline{v}_N = \frac{dr}{dt}\underline{e}_r \quad - (5)$$

The total velocity $\underline{v}$ is the sum of $\underline{v}_N$

2) and a velocity due to the rotation of the frame of reference.

Newtonian dynamics applies only to a frame of reference that is not moving.

The unit vectors of the plane polar coordinate system are:

$$\underline{k} \times \underline{e}_r = \underline{e}_\theta$$
$$\underline{e}_\theta \times \underline{k} = \underline{e}_r \qquad -(6)$$
$$\underline{e}_r \times \underline{e}_\theta = \underline{k}$$

The angular velocity is defined by:

$$\underline{\omega} = \omega \underline{k} \qquad -(7)$$

From eqns (4) to (7):

$$\underline{v} = \underline{v}_N + \omega \underline{k} \times r \underline{e}_r \qquad -(8)$$

i.e.

$$\boxed{\underline{v} = \underline{v}_N + \underline{\omega} \times \underline{r}} \qquad -(9)$$

the term $\underline{\omega} \times \underline{r}$ is the velocity due to the rotation of the frame of reference. Note carefully that this is absent completely for Newtonian dynamics.

3)

Therefore:

$$\underline{V} = \frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta$$

$$= \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r}$$ — (10)

$$= \underline{V}_N + \underline{\omega} \times \underline{r}$$

The complete velocity can be described by the Lagrangian dynamics, but not by Newtonian dynamics.

The lagrangian is:

$$\mathcal{L} = \frac{1}{2}mv^2 - V$$ — (11)

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$ — (12)

$$= \left(\frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta\right) \cdot \left(\frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta\right)$$

Clearly:

$$v^2 = V_N^2 + (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r})$$ — (13)

In vector analysis:

$$(\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) = (\underline{A} \cdot \underline{C})(\underline{B} \cdot \underline{D}) - (\underline{A} \cdot \underline{D})(\underline{B} \cdot \underline{C})$$

— (14)

4) So

$$(\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r}) = \omega^2 r^2 - \omega^2 r^2 = 0 \quad - (15)$$

and

$$v^2 = v_N^2. \quad - (16)$$

This result means that:

$$v_x^2 + v_y^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2. \quad - (17)$$

It is very important to note that eq. (10) is a covariant derivative:

$$\boxed{\underline{v} = \frac{D\underline{r}}{dt} = \left(\frac{d\underline{r}}{dt}\right)_N + \underline{\omega} \times \underline{r} \quad - (18)}$$

where the spin convective term is $\underline{\omega} \times \underline{r}$, due to the rotation of the axes. In eq. (18):

$$\left(\frac{d\underline{r}}{dt}\right)_N = \frac{dr}{dt} \underline{e}_r, \quad - (19)$$

and

$$\underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta. \quad - (20)$$

so

$$\underline{v} = \frac{D\underline{r}}{dt} = \frac{dr}{dt} \underline{e}_r + \omega r \underline{e}_\theta \quad - (21)$$

5)

Eqs. (18) and (21) are true for any vector $\underline{V}$, because the definition of the covariant derivative is true for any vector $\underline{V}$. So:

$$\frac{D\underline{V}}{dt} = \left(\frac{d\underline{V}}{dt}\right)_N + \underline{\omega} \times \underline{V} \qquad -(22)$$

$$= \frac{dV}{dt}\underline{e}_r + \omega V \underline{e}_\theta$$

if:

$$\underline{V} = V\underline{e}_r. \qquad -(23)$$

Eq. (22) is a special case of:

$$D_\mu V^a = \partial_\mu V^a + \omega^a{}_{\mu b} V^{\tilde{}} \qquad -(24)$$

of Cartan geometry. This means that the Cartan torsion is present in all of dynamics.

Einsteinian general relativity is therefore fundamentally incorrect because it uses zero torsion.

From eq. (2) the acceleration is defined as:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt}\right) \qquad -(25)$$

6) i.e. :

$$\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \frac{dr}{dt}\frac{d\underline{e}_r}{dt} + \frac{dr}{dt}\frac{d\underline{e}_r}{dt} + r\frac{d^2\underline{e}_r}{dt^2} \quad -(26)$$

i.e

$$\boxed{\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \left(2\frac{dr}{dt}\frac{d\underline{e}_r}{dt} + r\frac{d^2\underline{e}_r}{dt^2}\right) \quad -(27)}$$

The term in brackets is the acceleration due to the rotating frame itself. It is therefore due to the spin convection.

From fundamentals:

$$\frac{d\underline{e}_r}{dt} = \frac{d\theta}{dt}\underline{e}_\theta = \omega \underline{e}_\theta, \quad -(28)$$

so:

$$\underline{a} = \frac{d^2 r}{dt^2} + \left(2\frac{dr}{dt}\omega\underline{e}_\theta + r\frac{d}{dt}\left(\omega\underline{e}_\theta\right)\right) \quad -(29)$$

Now we:

$$\omega\underline{e}_\theta = \underline{\omega} \times \underline{e}_r \quad -(30)$$

and

$$\underline{V}_N = \frac{dr}{dt}\underline{e}_r \quad -(31)$$

to find that:

7)

$$\underline{a} = \frac{d\underline{v}_N}{dt} + 2\underline{\omega} \times \underline{v}_N + \omega r \frac{d\underline{e}_\theta}{dt} \quad -(32)$$

Finally use:

$$\frac{d\underline{e}_\theta}{dt} = -\omega\underline{e}_r \quad -(33)$$

for the fundamentals. Therefore:

$$\underline{a} = \frac{d\underline{v}_N}{dt} + 2\underline{\omega} \times \underline{v}_N - \omega^2 r \underline{e}_r, \quad -(34)$$

in which:

$$\underline{\omega} \times \underline{r} = \omega \underline{k} \times r\underline{e}_r = \omega r \underline{e}_\theta \quad -(35)$$

and

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \omega^2 r \underline{k} \times \underline{e}_\theta$$
$$= -\omega^2 r \underline{e}_r, \quad -(36)$$

So

$$\boxed{\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \underline{e}_r} \quad -(37)$$

and

$$\boxed{\underline{a} = \frac{d\underline{v}_N}{dt} + 2\underline{\omega} \times \underline{v}_N + \underline{\omega} \times (\underline{\omega} \times \underline{r})}$$

$$-(38)$$

This equation is again a covariant derivative.

8):

$$\underline{a} = \frac{D\underline{v}}{dt} = \frac{d\underline{v}_N}{dt} + 2\underline{\omega} \times \underline{v}_N + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$
$$- (39)$$

in which the spin connection operator is $\underline{\omega} \times$.
The three accelerations are the Newtonian, Coriolis and centrifugal. The Coriolis and centrifugal accelerations are defined by the rotating frame and by the spin connection.

Neither Hooke nor Newton understood the meaning of the Coriolis and centrifugal accelerations. Coriolis derived them in the nineteenth century but Cartan geometry was not available until the nineteen twenties. In this note they are understood for the first time as manifestations of the spin connection.

As in the preceding note, eqn. (39) can also be derived from:

$$\underline{v} = \underline{v}_N + \underline{\omega} \times \underline{r} \quad - (40)$$

$$\underline{a}_1 = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} \quad - (41)$$

Therefore:

9)

$$\underline{a}_1 = \frac{d}{dt}\left(\underline{v}_N + \underline{\omega} \times \underline{r}\right) + \underline{\omega} \times \left(\underline{v}_N + \underline{\omega} \times \underline{r}\right) - (42)$$

$$= \frac{d\underline{v}_N}{dt} + \underline{\omega} \times \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{v}_N + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right)$$

where

$$\frac{d\underline{r}}{dt} = \underline{v}_N + \underline{\omega} \times \underline{r} - (43)$$

so:

$$\underline{a}_1 = \frac{d\underline{v}_N}{dt} + 2\underline{\omega} \times \underline{v}_N + 2\underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) - (44).$$

Note that the centrifugal part of $\underline{a}_1$ is twice the centrifugal part of $\underline{a}$. So:

$$\underline{a}_1 = \underline{a} + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) - (45)$$

This is a matter of definition. The result (39) is derived from:

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

$$= \frac{d\underline{v}_N}{dt} + 2\underline{\omega} \times \underline{v}_N + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right)$$

$$= \ddot{r}\,\underline{e}_r - r\dot{\theta}^2\,\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

$$- (46)$$

10) The terms due to the spin connection are:

$$\frac{d}{dt}\left(r\dot{\theta}\underline{e}_\theta\right) + \ddot{r}\underline{e}_r = 2\underline{\omega}\times\underline{v}_N + \underline{\omega}\times\left(\underline{\omega}\times\underline{r}\right) \quad -(47)$$

$$= -\dot{\theta}^2 r\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

where

$$\underline{\omega}\times\left(\underline{\omega}\times\underline{r}\right) = -\dot{\theta}^2 r\underline{e}_r \quad -(48)$$

$$\omega = \dot{\theta} \quad -(49)$$

and

Therefore the Coriolis term is:

$$\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta = 2\underline{\omega}\times\underline{v}_N \quad -(50)$$

when $\underline{\omega}$ is constant, i.e

$$\frac{d\omega}{dt} = \ddot{\theta} = 0 \quad -(51)$$

In the more general case when $\underline{\omega}$ is not independent of time:

$$\frac{d}{dt}\left(\frac{d\underline{e}_r}{dt}\right) = \frac{d}{dt}\left(\frac{d\theta}{dt}\underline{e}_\theta\right)$$

$$= \omega\frac{d\underline{e}_\theta}{dt} + \frac{d\omega}{dt}\underline{e}_\theta \quad -(52)$$

$$= \omega\frac{d\underline{e}_\theta}{dt} + \dot{\omega}\underline{e}_\theta$$

11) so eq. (29) becomes :

$$\underline{a} = \frac{d^2 r}{dt^2} + 2 \frac{dr}{dt} \omega \underline{e}_\theta + \omega r \frac{d\underline{e}_\theta}{dt} + \dot{\omega} r \underline{e}_\theta$$

$$- (53)$$

The **centrifugal** acceleration is :

$$\boxed{\omega r \frac{d\underline{e}_\theta}{dt} = -\dot{\theta}^2 r \underline{e}_r = \underline{\omega} \times (\underline{\omega} \times \underline{r})}$$

$$- (54)$$

The ~~centripetal~~ **Coriolis** acceleration is

$$\left(2 \frac{dr}{dt} \omega + \dot{\omega} r \right) \underline{e}_\theta = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) \underline{e}_\theta$$

$$- (55)$$

where

$$\underline{e}_\theta = \underline{k} \times \underline{e}_r . \quad - (56)$$

So if the angular velocity is time dependent, the
Coriolis acceleration is :

$$\boxed{\begin{aligned} \underline{a}_{COR} &= 2\underline{\omega} \times \underline{v}_N + r \frac{d\omega}{dt} \underline{k} \times \underline{e}_r \\ &= 2\underline{\omega} \times \underline{v}_N + \frac{d\underline{\omega}}{dt} \times \underline{r} \\ &= \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) \underline{e}_\theta \qquad - (57) \end{aligned}}$$

12)

In previous work it was found that for all orbits in spherical spacetime:

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta = \underline{0} \quad -(58)$$

So the Coriolis acceleration of all orbit vanishes

In orbital theory:

$$\underline{a}_{COR} = 2\underline{\omega} \times \underline{v}_N + \frac{d\underline{\omega}}{dt} \times \underline{r} \quad -(59)$$

$$= (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta = \underline{0}$$

and the total acceleration of the orbit is:

$$\boxed{\begin{array}{l} \underline{a} = \left(\dfrac{d^2r}{dt^2}\right)\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \\[4mm] \quad = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r \end{array}} \quad -(60)$$

The centrifugal part of this acceleration is:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r\,\underline{e}_r \quad -(61)$$

and is a positive acceleration due to the spin connection.