234(10): Tetrads of the Minkowski Metric.

The infinitesimal line element and metric are related by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad -(1)$$

and the Minkowski metric is:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(2)$$

in the Cartesian basis.

Denote the metric in the circular polar basis by:

$$g_{\mu\nu} = q_\mu^{(a)} q_\nu^{(b)} \eta_{(b)(a)} \quad -(3)$$

where $q_\mu^{(a)}$ and $q_\nu^{(b)}$ are Cartan tetrads. The unit vectors of the circular polar basis are:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad -(4)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad -(5)$$

and the tetrad is defined in general by:

$$V^{(a)} = q_\mu^{(a)} V^\mu . \quad -(6)$$

It follows that:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(7)$$

for the transverse components. So the unit vectors of the two bases are related by the tetrad components:

2)

$$\underline{e}^{(1)} = q_1^{(1)} \underline{i} + q_2^{(1)} \underline{j} \quad -(8)$$

$$\underline{e}^{(2)} = q_1^{(2)} \underline{i} + q_2^{(2)} \underline{j}. \quad -(9)$$

The position vector in the two bases is defined in a plane by:

$$\underset{\sim}{r} = X \underline{i} + Y \underline{j} \quad -(10)$$

$$= r^{(2)} \underline{e}^{(1)} + r^{(1)} \underline{e}^{(2)}$$

The complex conjugate of $\underline{r}$ is defined by:

$$\underset{\sim}{r}^* = r^{(1)} \underline{e}^{(2)} + r^{(2)} \underline{e}^{(1)}. \quad -(11)$$

From eq. (10): $r^2 = X^2 + Y^2. \quad -(12)$

In the complex circular basis:

$$\underset{\sim}{r} \cdot \underline{r}^* = r^{(1)} r^{(2)} \underline{e}^{(1)} \cdot \underline{e}^{(2)} + r^{(2)} r^{(1)} \underline{e}^{(2)} \cdot \underline{e}^{(1)} \quad -(13)$$

$$+ r^{(1)2} \underline{e}^{(1)} \cdot \underline{e}^{(1)} + r^{(2)2} \underline{e}^{(2)} \cdot \underline{e}^{(2)}$$

$$= r^{(1)} r^{(2)} + r^{(2)} r^{(1)}$$

because $\underline{e}^{(1)} \cdot \underline{e}^{(2)} = \underline{e}^{(2)} \cdot \underline{e}^{(1)} = 1 \quad -(14)$

and $\underline{e}^{(1)} \cdot \underline{e}^{(1)} = \underline{e}^{(2)} \cdot \underline{e}^{(2)} = 0 \quad -(15)$

Therefore $X^2 + Y^2 = r^{(1)} r^{(2)} + r^{(2)} r^{(1)}. \quad -(16)$

From eq. (6) it follows that:

3)
$$\begin{bmatrix} r^{(1)} \\ r^{(2)} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad -(17)$$

where
$$q_1^{(1)} = \frac{1}{\sqrt{2}}, \quad q_2^{(1)} = -\frac{i}{\sqrt{2}}, \quad -(18)$$
$$q_1^{(2)} = \frac{1}{\sqrt{2}}, \quad q_2^{(2)} = \frac{i}{\sqrt{2}},$$

So
$$\boxed{r^{(1)} = \frac{1}{\sqrt{2}}(X - iY); \quad r^{(2)} = \frac{1}{\sqrt{2}}(X + iY)} \quad -(19)$$

It follows that:
$$\underline{r} = r^{(2)} \underline{e}^{(1)} + r^{(1)} \underline{e}^{(2)}$$
$$= \frac{1}{\sqrt{2}}(X + iY)\frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) + \frac{1}{\sqrt{2}}(X - iY)\frac{1}{\sqrt{2}}(\underline{i} + i\underline{j})$$
$$= X\underline{i} + Y\underline{j} \quad -(20)$$

QED. The complex circular basis describes circularly polarized radiation when a phase is added.

From the basic definition:
$$\underline{r} = X\underline{i} + Y\underline{j}, \quad -(21)$$

then
$$\frac{\partial \underline{r}}{\partial X} = \underline{i}, \quad \frac{\partial \underline{r}}{\partial Y} = \underline{j} \quad -(22)$$

4) and $\underline{dr} = \left(\frac{\underline{dr}}{dX}\right) dX + \left(\frac{\underline{dr}}{dY}\right) dY$. — (23)

$$= \underline{i}\, dX + \underline{j}\, dY$$

So: $ds^2 = \underline{dr} \cdot \underline{dr} = dX^2 + dY^2$ — (24)

and for two dimensional space, from eq. (1):

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{— (25)}$$

in the Cartesian basis.

In the complex circular basis:

$$\underset{\sim}{r} = r^{(2)} \underline{e}^{(1)} + r^{(1)} \underline{e}^{(2)} \quad \text{— (26)}$$

So $\underline{e}^{(1)} = \frac{\underline{dr}}{dr^{(2)}}, \quad \underline{e}^{(2)} = \frac{\underline{dr}}{dr^{(1)}}$, — (27)

and $\underline{dr} = \frac{\underline{dr}}{dr^{(2)}} dr^{(2)} + \frac{\underline{dr}}{dr^{(1)}} dr^{(1)}$ — (28)

$$= \underline{e}^{(1)} dr^{(2)} + \underline{e}^{(2)} dr^{(1)}.$$

So $\boxed{\begin{aligned} ds^2 &= \underline{dr} \cdot \underline{dr}^* \\ &= dr^{(2)} dr^{(1)} + dr^{(1)} dr^{(2)} \end{aligned}}$ — (29)

5)

Therefore:

$$ds^2 = \begin{bmatrix} dX & dY \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dX \\ dY \end{bmatrix} \quad -(30)$$

$$= \begin{bmatrix} dr^{(2)} & dr^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dr^{(1)} \\ dr^{(2)} \end{bmatrix} .$$

It follows that:

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(31)$$

and

$$\eta_{(a)(b)} = \begin{bmatrix} \eta_{(2)(1)} & 0 \\ 0 & \eta_{(1)(2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(32) .$$

The two metrics represent the same plane in two different ways. They are related by: $-(33)$

$$g_{\mu\nu} = q^{(b)}_{\mu} q^{(a)}_{\nu} \eta_{(a)(b)} \quad -(34)$$

i.e.

$$g_{11} = q^{(1)}_{1} q^{(2)}_{1} \eta_{(2)(1)} + q^{(2)}_{1} q^{(1)}_{1} \eta_{(1)(2)}$$

$$g_{22} = q^{(1)}_{2} q^{(2)}_{2} \eta_{(2)(1)} + q^{(2)}_{2} q^{(1)}_{2} \eta_{(1)(2)}$$

$-(35)$

6) where the tetrad elements are given by eq. (18), and the metric elements by eqs. (31) and (32).

It is important to note that eqs. (33) to (35) are still valid when a *phase* is incorporated, for example the phase of a plane wave propagating in the Z axis:

$$\phi = \omega t - \kappa Z \quad -(36)$$

where ω is the angular frequency and κ the wave vector magnitude. In this case:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{i\phi} \quad -(37)$$

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{-i\phi} \quad -(38)$$

so

$$q^{(1)}_1 = \frac{1}{\sqrt{2}} e^{i\phi}; \quad q^{(2)}_1 = \frac{1}{\sqrt{2}} e^{-i\phi},$$

$$q^{(2)}_1 = -\frac{i}{\sqrt{2}} e^{i\phi}, \quad q^{(2)}_2 = \frac{i}{\sqrt{2}} e^{-i\phi}$$

$$-(39)$$

and eqs (33) to (35) are still true.

Therefore the Minkowski space is definable by the Cartan tetrads with phase.

7) The complete Minkowski metric is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(40)$$

with the condition of:

$$q^{(0)}_{0} = q^{(3)}_{3} = 1. \quad -(41)$$

In ECE theory the electromagnetic potential is:

$$A^{a}_{\mu} = A_0 \, q^{a}_{\mu} \quad -(42)$$

where A_0 is a potential amplitude. In eq. (42) the a index represents the complex circular basis, the natural one for circularly polarized radiation. The bracket around a has been omitted for ease of notation.

The Cartan torsion is then:

$$T^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \omega^{a}_{\nu b} q^{b}_{\mu} \quad -(43)$$

$$= \partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu\nu} - \omega^{a}_{\nu\mu}$$

The tetrad postulate is:

$$D_\mu q^{a}_{\nu} = \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \Gamma^{\lambda}_{\mu\nu} q^{a}_{\lambda} = 0 \quad -(44)$$

8)

i.e. $$\partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} = 0. \quad -(45)$$

Therefore: $$\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu}. \quad -(46)$$

The Minkowski spacetime may therefore be associated with a connection:

$$\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} = \partial_\mu q^a_\nu. \quad -(47)$$

For example:

$$\Gamma^{(1)}_{31} - \omega^{(1)}_{31} = \partial_3 q^{(1)}_1$$

$$= \frac{1}{\sqrt{2}} \frac{\partial}{\partial Z} e^{i(\omega t - \kappa Z)}$$

$$= -\frac{i\kappa}{\sqrt{2}} e^{i(\omega t - \kappa Z)} \quad -(48)$$

$$\neq 0.$$

The real part of this connection is:

$$\boxed{\mathrm{Re}\left(\Gamma^{(1)}_{31} - \omega^{(1)}_{31}\right) = \frac{\kappa}{\sqrt{2}} \sin(\omega t - \kappa Z)} \quad -(49)$$

We name this the <u>dynamic connection</u> and denote it:

$$\boxed{\varsigma^{a}_{\mu\nu} := \Gamma^{a}_{\mu\nu} - \omega^{a}_{\mu\nu}} \quad -(50)$$

i.e.

$$\boxed{\varsigma^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu}} \quad -(51)$$

When the phase is zero the dynamic connection vanishes, and the analysis reduces to the usual interpretation of the Minkowski metric as leaving no connection.

The significance of the dynamic connection to cosmology is that it allows the field equation of gravitation to be derived from the metric in a generally covariant manner. The field equations are based on the notion of spacetime and include the connection.

Therefore all of cosmology can be based on the Minkowski metric.