234(3): Minkowski Metric; Relativistic Dynamics.

The relativistic dynamics mean that time derivatives are derivatives with respect to proper time. Therefore:

$$\underline{r} = r \, \underline{e}_r \quad -(1)$$

$$\underline{v} = \left(\frac{dr}{d\tau}\right) \underline{e}_r + r \left(\frac{d\theta}{d\tau}\right) \underline{e}_\theta . \quad -(2)$$

The unit vectors obey the equations:

$$\frac{d\underline{e}_r}{d\tau} = \left(\frac{d\theta}{d\tau}\right) \underline{e}_\theta \quad -(3)$$

$$\frac{d\underline{e}_\theta}{d\tau} = -\left(\frac{d\theta}{d\tau}\right) \underline{e}_r . \quad -(4)$$

The acceleration is ~~therefore~~:

$$\underline{a} = \frac{d\underline{v}}{d\tau} = \frac{d}{d\tau}\left(\left(\frac{dr}{d\tau}\right)\underline{e}_r + r\left(\frac{d\theta}{d\tau}\right)\underline{e}_\theta\right)$$

$$= \frac{d}{d\tau}\left(\frac{dr}{d\tau}\right)\underline{e}_r + \left(\frac{dr}{d\tau}\right)\left(\frac{d\underline{e}_r}{d\tau}\right) \quad -(5)$$
$$+ \left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)\underline{e}_\theta + r\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right)\underline{e}_\theta + r\left(\frac{d\theta}{d\tau}\right)\left(\frac{d\underline{e}_\theta}{d\tau}\right)$$

Now use eqs. (3) and (4) in eq. (5):

2)

$$\underline{a} = \left(\frac{d^2 r}{d\tau^2}\right)\underline{e}_r + \left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)\underline{e}_\theta + \left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)\underline{e}_\theta + r\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right)\underline{e}_\theta - r\left(\frac{d\theta}{d\tau}\right)^2\underline{e}_r$$

$$= \left(\frac{d^2 r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2\right)\underline{e}_r + \left(r\frac{d^2\theta}{d\tau^2} + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)\right)\underline{e}_\theta \qquad -(6)$$

From note 234(2):

$$\frac{d^2 r}{d\tau^2} \neq 0 \qquad -(7)$$

but

$$\underline{p} = m\frac{dr}{d\tau} \qquad -(8)$$

is a constant of motion. So: $\qquad -(9)$

$$\boxed{\underline{a} = \left(\frac{1}{m}\frac{dp}{d\tau} - r\left(\frac{d\theta}{d\tau}\right)^2\right)\underline{e}_r + \left(r\frac{d^2\theta}{d\tau^2} + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)\right)\underline{e}_\theta}$$

The <u>Minkowski force</u> is:

$$\underline{F} = m\underline{a} \qquad -(10)$$

3)

Here:

$$L = mr^2 \frac{d\theta}{d\tau} \quad -(11)$$

and

$$p = m \frac{dr}{d\tau} \quad -(12)$$

and

$$E = mc^2 \frac{dt}{d\tau}. \quad -(13)$$

In the Minkowski theory of cosmology, E, p and L are all constants of motion.

So:

$$\left(\frac{d\theta}{d\tau}\right)^2 = \left(\frac{L}{mr^2}\right)^2 \quad -(14)$$

$$\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) = \frac{L}{m}\frac{d}{d\tau}\left(\frac{1}{r^2}\right) \quad -(15)$$

$$\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) = \frac{p}{m}\cdot\frac{L}{mr^2} = \left(\frac{pL}{m^2}\right)\frac{1}{r^2} \quad -(16)$$

So:

$$\underline{a} = -\left(\frac{L}{m}\right)^2 \frac{1}{r^3}\underline{e}_r + \frac{1}{m}\frac{dp}{d\tau}\underline{e}_r \quad -(17)$$
$$+ \left(\frac{Lr}{m}\frac{d}{d\tau}\left(\frac{1}{r^2}\right) + \frac{2p\cdot L}{m^2}\frac{1}{r^2}\right)\underline{e}_\theta$$

Finally use:

4) $$\frac{df}{d\tau} = \frac{df}{dr}\frac{dr}{d\tau}, \quad f = \frac{1}{r^2} \qquad -(18)$$

So $$\frac{d}{d\tau}\left(\frac{1}{r^2}\right) = -\frac{2}{r^3}\frac{dr}{d\tau} = -\frac{2p}{mr^3} \qquad -(19)$$

So:
$$\underline{a} = -\left(\frac{L}{m}\right)^2 \frac{1}{r^3}\underline{e}_r + \frac{1}{m}\frac{dp}{d\tau}\underline{e}_r + \left(\frac{2pL}{m^2}\frac{1}{r^2} - \frac{2pL}{m^2 r^2}\right)\underline{e}_\theta \qquad -(20)$$

i.e.
$$\boxed{\underline{a} = \left[\frac{1}{m}\frac{dp}{d\tau} - \left(\frac{L}{m}\right)^2 \frac{1}{r^3}\right]\underline{e}_r} \qquad -(21)$$

The Minkowski force is:
$$\boxed{\underline{F} = \left(\frac{dp}{d\tau} - \frac{L^2}{mr^3}\right)\underline{e}_r} \qquad -(22)$$

<u>Conclusion</u>

The force (22) is equivalent to the infinitesimal line element:
$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \qquad -(23)$$

Eq. (22) is the acceleration (21) multiplied