234(5): The General Expression for Force for Any Orbit.

From the kinematics of note 234(3), the acceleration generated by any orbit is:

$$\underline{a} = \left(\frac{d^2 r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2\right)\underline{e}_r + \left(r\frac{d^2\theta}{d\tau^2} + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)\right)\underline{e}_\theta \quad -(1)$$

in plane polar coordinates.

For the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(2)$$

and

$$L = mr^2 \frac{d\theta}{d\tau} \quad -(3)$$

So:

$$r\frac{d^2\theta}{d\tau^2} + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)$$

$$= r\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)$$

$$= \frac{Lr}{m}\frac{d}{d\tau}\left(\frac{1}{r^2}\right) + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)$$

$$= -\frac{2L}{mr^2}\left(\frac{dr}{d\tau}\right) + \frac{2L}{mr^2}\left(\frac{dr}{d\tau}\right) \quad -(3)$$

$$= 0.$$

This result is also true for a metric:

$$ds^2 = c^2 d\tau^2 = Ac^2 dt^2 - B dr^2 - r^2 d\theta^2 \quad -(4)$$

2) For the Carter metric:

$$L = m\,C(r)\frac{d\theta}{d\tau} \quad -(5)$$

so

$$\frac{d\theta}{d\tau} = \frac{L}{m\,C(r)} \quad -(6)$$

and

$$r\frac{d^2\theta}{d\tau^2} + 2\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) \quad -(7)$$

$$= \frac{rL}{m}\frac{d}{d\tau}\left(\frac{1}{C(r)}\right) + 2\left(\frac{dr}{d\tau}\right)\frac{L}{m\,C(r)}$$

and in this case there is a component of $\underline{a}$ in $\underline{e}_\theta$.

For the Minkowski metric:

$$\boxed{\underline{a} = \left(\frac{d^2r}{d\tau^2} - \frac{L^2}{m^2r^3}\right)\underline{e}_r} \quad -(8)$$

which the is the acceleration for any observable orbit:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}\right) \quad -(9)$$

with

$$p = \gamma m v \quad -(10)$$

$$L = \gamma m r^2 \omega. \quad -(11)$$

In eq. (8):

3)

$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\frac{d\theta}{d\tau} \quad -(12)$$

where

$$\frac{d\theta}{d\tau} = \frac{L}{mr^2} \quad -(13)$$

So:

$$\frac{dr}{d\tau} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad -(14)$$

where $dr/d\theta$ is always given by eq. (9)

Denote

$$f = \frac{dr}{d\tau} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad -(15)$$

then

$$\frac{df}{d\tau} = \frac{df}{dr}\frac{dr}{d\tau} = \frac{df}{dr}\left(\frac{L}{mr^2}\right)\left(\frac{dr}{d\theta}\right) \quad -(16)$$

i.e

$$\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) = \frac{L^2}{m^2r^2}\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) \quad -(17)$$

So the acceleration for any observable orbit $dr/d\theta$ in the universe is:

$$\boxed{\underline{a} = \left(\frac{L^2}{m^2r^2}\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{L^2}{m^2r^3}\right)\underline{e}_r} \quad -(18)$$

4) If the orbit is the ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(19)$$

then

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(20)$$

and:

$$\underline{a} = \left(\frac{L^2 \epsilon^2}{m^2 \alpha^2} \sin\theta \frac{d}{dr}\left(\sin\theta\right) - \frac{L^2}{m^2 r^3} \right) \underline{e}_r \quad -(21)$$

where

$$\frac{d\sin\theta}{dr} = \cos\theta \frac{d\theta}{dr} = \frac{\alpha\cos\theta}{\epsilon r^2 \sin\theta} \quad -(22)$$

so

$$\underline{a} = \left(\frac{L^2 \epsilon}{m^2 r^2 \alpha} \cos\theta - \frac{L^2}{m^2 r^3} \right) \underline{e}_r \quad -(23)$$

which is the result given in note 234(4) and UFT 196,

Q.E.D. So the analysis is completely self consistent.

For the ellipse:

$$\epsilon\cos\theta = \frac{\alpha}{r} - 1 \quad -(24)$$

so:

$$\underline{a} = \left(\frac{L^2}{m^2 r^3} - \frac{L^2}{m^2 r^2 \alpha} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r \quad -(25)$$

Eq. (25) can be written as:

5)

$$\underline{a} = -\frac{L^2}{m^2 r^2 \alpha} \underline{e}_r \quad -(26)$$

$$= -\frac{MG}{r^2} \underline{e}_r$$

If the Newtonian idea of orbital dynamics is introduced. In this idea:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(27)$$

and

$$E = \frac{p^2}{2m} - \frac{mMG}{r}. \quad -(28)$$

Finally the "force of attraction" is defined as:

$$\underline{F} = m\underline{a} = -\frac{mMG}{r^2} \underline{e}_r \quad -(29)$$

The result (29) is true if and only if α is defined as the orbit is an ellipse in which the result of the assumption eq. (27). Eq (27) is the (28). Note carefully that eq. (18) is the correct balance of accelerations, but the Newtonian result (29) is incorrect because there is only a "force of attraction."

6)

The correct way to interpret the result (26) is that the elliptical orbit results in an acceleration that is negative in sign and directed centrally, or radially. The orbit is the direct result of the metric. The most general description of the acceleration is given by eq. (18), which can be written as:

$$\underline{a} = \left(\frac{L}{mr}\right)^2 \left(\left(\frac{dr}{d\theta}\right) \frac{d}{dr} \left(\frac{1}{r^2} \frac{dr}{d\theta} \right) - \frac{1}{r} \right) \underline{e}_r \quad -(30)$$

and in general this does not give the inverse square law (29).

Eq. (30) will be worked out for various orbits and metrics in the next note.