234(4): Comparison of the New Minkowski Method with the Conventional Calculation of Force in Plane Polar Coordinates

The conventional calculation (UFT196) proceeds by

$$\underline{r} = r\,\underline{e}_r \qquad -(1)$$

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \qquad -(2)$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \qquad -(3)$$

These are expressions in plane polar coordinates of:

$$\underline{v} = \frac{d\underline{r}}{dt}, \quad \underline{a} = \frac{d\underline{v}}{dt}. \qquad -(4)$$

The unit vectors of the plane polar coordinate system are defined by:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \qquad -(5)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta. \qquad -(6)$$

This is a classical, non-relativistic, analysis in which the concept of potential energy is present from the outset. The lagrangian is:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \qquad -(7)$$

where the reduced mass is:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad -(8)$$

The mass m_1 is gravitationally attracted to the mass m_2.

2) The Lagrangian is cyclic in θ and the conjugate angular momentum is conserved. So the two Euler Lagrange equations are:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \quad -(9)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(10)$$

the angular momentum is conserved so:

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0. \quad -(11)$$

The angular momentum is defined as:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(12)$$

If $m_2 \gg m_1$ and $m_1 := m$ $\quad -(13)$

then from eqns. (7) and (12):

$$L = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt} \quad -(13)$$

From eq. (2):

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad -(14)$$

so

$$\mathcal{L} = \frac{1}{2}mv^2 - U(r). \quad -(15)$$

The Hamiltonian is:

$$H = \frac{1}{2} m v^2 + U(r). \quad -(16)$$

The radial part of the velocity is:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \omega \frac{dr}{d\theta} \quad -(17)$$

In Newtonian dynamics:

$$U(r) = -\frac{mMG}{r} \quad -(18)$$

where m is the mass of a planet attracted by the mass M of the sun. The hamiltonian is identified with the total classical energy:

$$H = E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r) \quad -(19)$$

Therefore

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{m}\left(E - U\right) - \frac{L^2}{m^2 r^2} \quad -(20)$$

Using eq. (17):

$$\frac{d\theta}{dr} = \frac{L}{r^2}\left(2m\left(E - U - \frac{L^2}{2mr^2}\right)\right)^{-1/2} \quad -(21)$$

This is the equation of an ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(22)$$

4) from which

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \quad -(23)$$

From eqs. (21) to (23):

$$\alpha = \frac{L^2}{m^2 M G} \quad -(24)$$

$$\epsilon = \left(1 - \frac{2 E L^2}{m^3 M^2 G^2} \right)^{1/2} \quad -(25)$$

From eqs. (17) and (23):

$$\frac{dr}{dt} = \left(\frac{L \epsilon}{m \alpha} \right) \sin\theta \quad -(26)$$

so

$$\dot{r} = \left(\frac{L\epsilon}{m\alpha} \right) \sin\theta, \quad \dot{\theta} = \frac{L}{m r^2}. \quad -(27)$$

In this analysis, dr/dt is not a constant of motion. In the relativistic Minkowski analysis of note 234(3) it is a constant of motion.

This illustrates the profound philosophical difference between Newtonian dynamics and relativistic dynamics. Eq. (10) shows that in Newtonian dynamics there exists a force defined by:

5)
$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \quad -(28)$$

which implies that:
$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m}{L^2} r^2 F(r) \quad -(29)$$

From eqs. (22) and (29):
$$F(r) = -\frac{mMG}{r^2} \quad -(30)$$

This is known as the inverse square law of Isaac Newton, but it was actually pointed out to Newton by Robert Hooke.

There is a fundamental problem in Newtonian dynamics because the force $F(r)$ is not counterbalanced, so m would be attracted into M, contrary to observation. The so called "centripetal potential" is in fact part of the kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} \quad -(31)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$= \frac{1}{2}m(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta)\cdot(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta)$$

In Newtonian dynamics the last term on the right hand side of eq. (31) is interpreted as the

) "centrifugal potential energy":

$$U_c = \frac{L^2}{2mr^2} \quad -(32)$$

obviously, this is not true, the term is part of the kinetic energy as in eq. (31), so no force can be defined to counterbalance the attractive force (30). The reason for this is that force is defined from potential energy, so

$$F_c \overset{?}{=} -\frac{\partial U_c}{\partial r} = \frac{L^2}{mr^3} \quad -(33)$$

is an incorrect definition.

It is easily shown as follows that Newtonian dynamics is incomplete because:

$$\ddot{r} = \left(\frac{L\epsilon}{m\alpha}\right)\frac{d\sin\theta}{dt}, \quad \ddot{\theta} = \frac{L}{m}\frac{d}{dt}\left(\frac{1}{r^2}\right) \quad -(34)$$

where:

$$\frac{d}{dt}\left(\frac{1}{r^2}\right) = -\frac{2}{r^3}\frac{dr}{dt}, \quad \frac{d\sin\theta}{dt} = \frac{d\theta}{dt}\cos\theta \quad -(35)$$

so

$$\ddot{r} = \left(\frac{\epsilon L^2}{m^2\alpha}\right)\frac{1}{r^2}\cos\theta, \quad \ddot{\theta} = -\left(\frac{2L^2\epsilon}{m^2\alpha}\right)\frac{\sin\theta}{r^3} \quad -(36)$$

and

$$\ddot{r} - r\dot{\theta}^2 = \frac{\epsilon MG}{r^2}\cos\theta - \frac{L^2}{m^2r^3} \quad -(37)$$

7) From eq. (3) the complete force is:

$$\underline{F} = m\underline{a} = m\left(\left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta\right) \quad -(38)$$

but from the Newtonian eq. (28) it is:

$$F = \left(\ddot{r} - r\dot{\theta}^2\right)m = -\frac{\partial U}{\partial r}. \quad -(39)$$

If by observation the orbit of a planet is an ellipse, then from eqs. (36):

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(40)$$

However for the general orbit, eq. (40) is not true and Newtonian dynamics fails.

From eq. (37):

$$\underline{F} = m\left(\frac{\epsilon MG}{r^2}\cos\theta - \frac{L^2}{m^2 r^3}\right)\underline{e}_r \quad -(41)$$

This does not look very much like eq. (30), and for the general orbit eq. (41) does not reduce to eq. (30), for example the orbits of stars in galaxies. For the ellipse however:

$$\epsilon\cos\theta = \frac{\alpha}{r} - 1 \quad -(42)$$

where

$$\alpha = \frac{L^2}{m^2 M G} \quad -(43)$$

So:

$$\underline{F} = \left(-\frac{mMG}{r^2} + \frac{L^2}{mr^3} - \frac{L^2}{mr^3} \right) \underline{e}_r$$

$$= -\frac{mMG}{r^2} \underline{e}_r \quad -(44)$$

The usual Newtonian point of view is that the net force on a particle in orbit is zero:

$$F \overset{?}{=} -\frac{mMG}{r^2} + \frac{L^2}{mr^3} = 0 \quad -(45)$$

because the attractive force with the minus sign is countered by the repulsive force with the positive sign. This is completely incorrect because the purely kinematic analysis given here results in eq. (44). The only assumption made is based on the observation of an elliptical orbit. Nothing else is assumed except for eq. (4).

The overall result therefore is that classical kinematics must be rejected in favour of a relativistic approach, the simplest of which is the one based on the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(46)$$

giving the Minkowski force:

$$\underline{F}_m = -\frac{L^2}{mr^3}\,\underline{e}_r \quad -(47)$$

as in note 234(3).

The origin of the relativistic Minkowski force is obviously the infinitesimal line element (46), which produces the orbit:

$$\frac{dr}{d\theta} = r^2\left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}\right)^{1/2} \quad -(48)$$

in which

$$\frac{d^2 r}{dt^2} = 0, \quad -(49)$$

because

$$\frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) = \underline{0}. \quad -(50)$$

Therefore in the Minkowski cosmology developed here for the first time, there is no radial or central force between m and M. The origin of the force (47) is the orbit itself, which is a property of spacetime. The orbit is the path along which the particle of mass m travels.

10) These ideas were introduced during the development of EGR, but using the incorrect:

$$ds^2 =? \; c^2 dt^2 \left(1 - \frac{r_0}{r}\right) - dr^2 \left(1 - \frac{r_0}{r}\right)^{-1} - r^2 d\theta^2, \quad -(51)$$

which is not only incorrect for many reasons, but needlessly complicated.

The correct way to interpret the force $\underline{F}$ in is that it is purely a definition, it is m multiplied by the acceleration:

$$\underline{a} = \frac{d\underline{v}}{dt} = -\frac{L^2}{m^2 r^3} \underline{e}_r \quad -(52).$$

produced by the infinitesimal line element (46).

Note carefully that the acceleration (52) is true for all orbits, not just an ellipse. Eq. (52) indicates that the change in velocity of a mass m in the orbit (48) is negative.