232(7): Deflection of Light with Precessing Hyperbola

The deflection is measured by the angle between the asymptotes:

$$\Delta\phi = 2\sin^{-1}\frac{1}{\epsilon} = 2\tan^{-1}\frac{b}{a} \quad -(1)$$

where the eccentricity is:

$$\epsilon = \left(1 + \frac{b^2}{a^2}\right)^{1/2} \quad -(2)$$

The Newtonian trajectory is defined by

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(3)$$

where

$$\alpha = a\left(\epsilon^2 - 1\right) \quad -(4)$$

From eq. (3):

$$\epsilon = \frac{1}{\cos\theta}\left(\frac{\alpha}{r} - 1\right) \quad -(5)$$

so:

$$\Delta\phi = 2\sin^{-1}\left(\left(\frac{\alpha}{r}-1\right)^{-1}\cos\theta\right) \quad -(6)$$

i.e.

$$\frac{1}{\epsilon} = \sin\left(\frac{\Delta\phi}{2}\right) = \cos\theta\left(\frac{\alpha}{r}-1\right)^{-1} \quad -(7)$$

so:

$$\boxed{\cos\theta = \left(\frac{\alpha}{r}-1\right)\sin\left(\frac{\Delta\phi}{2}\right)} \quad -(8)$$

2)

From the precession of the perihelia of planets it is known that:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(9)$$

which is also true for any conic section. Therefore in general:

$$\cos x\theta = \left(\frac{\alpha}{r}-1\right)\sin\left(\frac{\Delta\phi}{2}\right), \quad -(10)$$

so:

$$\boxed{x = \frac{1}{\theta}\cos^{-1}\left(\left(\frac{\alpha}{r}-1\right)\sin\left(\frac{\Delta\phi}{2}\right)\right)} \quad -(11)$$

for the deflection of a hyperbolic orbit by a mass M.

Therefore the deflection of light by gravity is defined by eq. (11). This uses the same theory as that describing perihelion precession in planets.

With reference to Fig. (1) the hyperbola is defined by the focus F. The mass M is at the focus, so

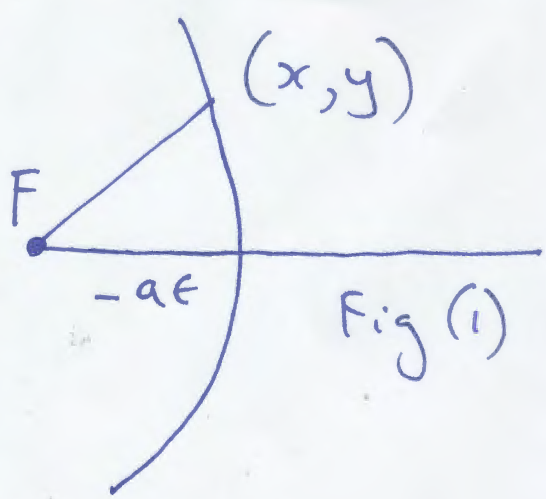

Fig (1)

3)

$$X = -a\epsilon + r\cos(x\theta) \quad -(12)$$

$$Y = r\sin(x\theta) \quad -(13)$$

$$r = -\epsilon X - a \quad -(14)$$

so eq. (9) follows. Also:

$$(X + a\epsilon)^2 + Y^2 = r^2 \quad -(15)$$

so:

$$X^2 + Y^2 + 2a\epsilon X + a^2\epsilon^2 = r^2 \quad -(16)$$

i.e.

$$X^2 + Y^2 + a^2\left(1 + \frac{b^2}{a^2}\right) = r^2 - 2a\epsilon X$$

$$= (\epsilon X + a)^2 - 2a\epsilon X$$

$$= \epsilon^2 X^2 + a^2 \quad -(17)$$

$$= \left(1 + \frac{b^2}{a^2}\right) X^2 + a^2$$

i.e.

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad -(18)$$

It follows that the well known equation (18) is not affected by:

$$\theta \rightarrow x\theta. \quad -(19)$$

The same is true for the ellipse:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1. \quad -(20)$$

4) If the angle of deflection is doubled then eq. (1) becomes

$$2\Delta\phi = 2\sin^{-1}\frac{1}{\epsilon_1} \quad -(21)$$

and

$$d_1 = a(\epsilon_1^2 - 1) \quad -(22)$$

therefore

$$\frac{1}{\epsilon} = \sin\left(\frac{\Delta\phi}{2}\right) \quad -(23)$$

$$\frac{1}{\epsilon_1} = \sin(\Delta\phi) \quad -(24)$$

For small deflections:

$$\frac{1}{\epsilon} = \frac{\Delta\phi}{2} \quad -(25)$$

$$\frac{1}{\epsilon_1} = \Delta\phi \quad -(26)$$

so

$$\frac{\epsilon_1}{\epsilon} = \frac{1}{2}. \quad -(27)$$

However, doubling the eccentricity can be thought of as a change in x, from x to x_1, so:

$$\frac{d_1}{1+\epsilon_1\cos(x\theta)} = \frac{d}{1+\epsilon\cos(x_1\theta)} \quad -(28)$$

where

$$d_1 = a(\epsilon_1^2 - 1) \quad -(29)$$

$$d = a(\epsilon^2 - 1) \quad -(30)$$

It follows that:

$$\frac{(\epsilon_1^2 - 1)}{1 + \epsilon_1 \cos(x\theta)} = \frac{(\epsilon^2 - 1)}{1 + \epsilon \cos(x_1 \theta)} \quad -(31)$$

where

$$\epsilon = 2\epsilon_1 \quad -(32)$$

So

$$\frac{(4\epsilon^2 - 1)}{1 + 2\epsilon \cos(x\theta)} = \frac{\epsilon^2 - 1}{1 + \epsilon \cos(x_1 \theta)} \quad -(33)$$

It follows that:

$$\cos(x_1 \theta) = \frac{1}{\epsilon}\left(\left(\frac{\epsilon^2 - 1}{4\epsilon^2 - 1}\right)\left(1 + 2\epsilon \cos(x\theta)\right) - 1\right) \quad -(34)$$

where:

$$-1 < \cos(x_1 \theta) < 1 \quad -(35)$$

and

$$-1 < \cos(x\theta) < 1 \quad -(36)$$

The requirements (35) and (36) restrict the possible values of x and x_1.