232(7): Deflection of Light with Precessing Hyperbola

The deflection is measured by the angle between the asymptotes:

$$\Delta\phi = 2\sin^{-1}\frac{1}{\epsilon} = 2\tan^{-1}\frac{b}{a} \quad -(1)$$

where the eccentricity is:

$$\epsilon = \left(1 + \frac{b^2}{a^2}\right)^{1/2} \quad -(2)$$

The Newtonian trajectory is defined by

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(3)$$

where

$$\alpha = a\left(\epsilon^2 - 1\right) \quad -(4)$$

From eq. (3):

$$\epsilon = \frac{1}{\cos\theta}\left(\frac{\alpha}{r} - 1\right) \quad -(5)$$

so:

$$\Delta\phi = 2\sin^{-1}\left(\left(\frac{\alpha}{r} - 1\right)^{-1}\cos\theta\right) \quad -(6)$$

i.e.

$$\frac{1}{\epsilon} = \sin\left(\frac{\Delta\phi}{2}\right) = \cos\theta\left(\frac{\alpha}{r} - 1\right)^{-1} \quad -(7)$$

so:

$$\boxed{\cos\theta = \left(\frac{\alpha}{r} - 1\right)\sin\left(\frac{\Delta\phi}{2}\right)} \quad -(8)$$

2)

From the precession of the perihelia of planets it is known that:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(9)$$

which is also true for any conic section. Therefore in general:

$$\cos x\theta = \left(\frac{\alpha}{r}-1\right)\sin\left(\frac{\Delta\phi}{2}\right), \quad -(10)$$

so:

$$\boxed{x = \frac{1}{\theta}\cos^{-1}\left(\left(\frac{\alpha}{r}-1\right)\sin\left(\frac{\Delta\phi}{2}\right)\right)} \quad -(11)$$

for the deflection of a hyperbolic orbit by a mass M.

Therefore the deflection of light by gravity is defined by eq. (11). This was the same theory as that describing perihelia precession in planets.

With reference to Fig.(1) the hyperbola is defined by the focus F. The mass M is at the focus, so

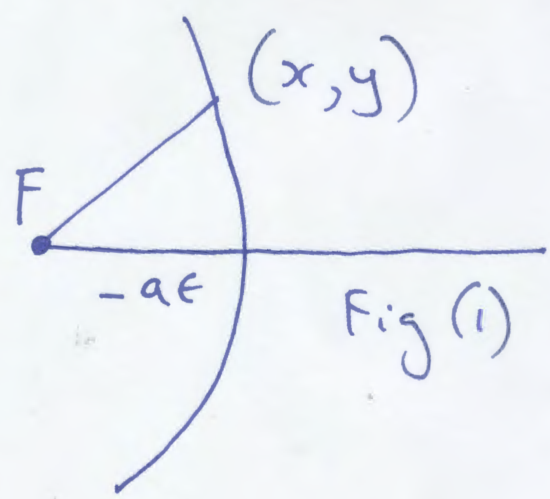

Fig (1)

3)

$$X = -a\epsilon + r\cos(x\theta) \quad -(12)$$
$$Y = r\sin(x\theta) \quad -(13)$$
$$r = -\epsilon X - a \quad -(14)$$

so eq. (9) follows. Also:

$$(X + a\epsilon)^2 + Y^2 = r^2 \quad -(15)$$

so:

$$X^2 + Y^2 + 2a\epsilon X + a^2\epsilon^2 = r^2 \quad -(16)$$

i.e.

$$X^2 + Y^2 + a^2\left(1 + \frac{b^2}{a^2}\right) = r^2 - 2a\epsilon X$$
$$= (\epsilon X + a)^2 - 2a\epsilon X$$
$$= \epsilon^2 X^2 + a^2 \quad -(17)$$
$$= \left(1 + \frac{b^2}{a^2}\right) X^2 + a^2$$

i.e.

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad -(18)$$

It follows that the well known equation (18) is not affected by:

$$\theta \rightarrow x\theta. \quad -(19)$$

The same is true for the ellipse:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad -(20)$$

4) and all conic or conical sections.

Consider now the experimental result:

$$\Delta\phi_1 = 2\Delta\phi \quad -(21)$$

Consider $\Delta\phi$ to be described by eq. (8), with

$$x = 1 \quad -(22)$$

then $\Delta\phi_1$ is described by:

$$\cos(x\theta) = \left(\frac{\alpha}{r} - 1\right)\sin\left(\frac{\Delta\phi_1}{2}\right)$$

$$= \left(\frac{\alpha}{r} - 1\right)\sin(\Delta\phi) \quad -(23)$$

$$-(24)$$

So:

$$\frac{\cos(x\theta)}{\cos\theta} = \frac{\sin(\Delta\phi)}{\sin(\Delta\phi/2)}$$

For very small deflections such as that of light by gravitation:

$$\sin\Delta\phi \sim \Delta\phi \quad -(25)$$

$$\sin\left(\frac{\Delta\phi}{2}\right) \sim \frac{\Delta\phi}{2} \quad -(26)$$

So

$$\cos(x\theta) = 2\cos\theta \quad -(27)$$

$$x = \frac{1}{\theta}\cos^{-1}(2\cos\theta) \quad -(28)$$

5) It follows that:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} = \frac{\alpha}{1 + 2\epsilon \cos\theta} \quad - (29)$$

This means that if a deflection of $\Delta\phi$ is assumed for the Newtonian (3) then a deflection of $2\Delta\phi$ is given by:

$$\epsilon \rightarrow 2\epsilon, \quad - (30)$$

the Newtonian (3) is changed to the Newtonian (29). The precession factor is given by eq. (28) and this can be graphed.

Therefore eq. (28) is an explanation of why deflection of light is twice the Newtonian value, given the general conical section (9).