232(4): Solution of the Equation of Motion of EGR with the Method of Variation of Constants.

The EGR equation of motion is the second order non-linear equation:

$$\frac{d^2u}{d\theta^2} + u - \frac{1}{\alpha} = \delta u^2 \quad -(1)$$

and has no known analytical solution. However, in the solar system $\delta / \alpha$ is a very small quantity so the equation can be solved approximately. The method of variation of constants is described in Stephenson, "Mathematical Methods for Science Students", pp. 376 ff.

Consider the second order constant coefficient linear equation:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x) \quad -(2)$$

Its reduced equation is:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0. \quad -(3)$$

Let $y_1$ and $y_2$ be two independent solutions of eq. (3). Then the complementary function:

$$y = A_1 y_1 + A_2 y_2 \quad -(4)$$

is also a solution of eq. (3).

Let the particular integral:

$$Y(x) = v_1(x) y_1 + v_2(x) y_2 \quad -(5)$$

be a solution of eq. (2).

2)

The general solution of eq. (2) is:

$$y = \left(A_1 + v_1(x)\right)y_1 + \left(A_2 + v_2(x)\right)y_2 \quad -(6)$$

It may be shown ~~that~~:

$$v_1(x) = -\frac{1}{a_0}\int \frac{y_2 f(x)}{y_1 y_2' - y_1' y_2}\,dx \quad -(7)$$

$$v_2(x) = \frac{1}{a_0}\int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}\,dx \quad -(8)$$

where $y_1 y_2' - y_1' y_2$ is the Wronskian.

For example, solve:

$$\frac{d^2y}{dx^2} - y = e^x \quad -(9)$$

The reduced equation is:

$$\frac{d^2y}{dx^2} - y = 0 \quad -(10)$$

The two solutions of eq. (10) are:

$$y_1 = e^x, \quad y_2 = e^{-x} \quad -(11)$$

The complementary function is:

$$y = A_1 e^x + A_2 e^{-x} \quad -(12)$$

The particular integral is:

3)

$$y(x) = v_1(x)e^{x} + v_2(x)e^{-x} \quad - (13)$$

So:

$$v_1(x) = -\int \frac{e^{-x}e^{x}\,dx}{(-e^{x}e^{-x} - e^{x}e^{-x})} = \frac{x}{2} \quad - (14)$$

$$v_2(x) = \int \frac{e^{x}e^{x}\,dx}{(-e^{x}e^{-x} - e^{x}e^{-x})} = -\frac{e^{2x}}{4} \quad - (15)$$

The general solution is therefore:

$$y = \left(A_1 + \frac{x}{2}\right)e^{x} + \left(A_2 - \frac{e^{2x}}{4}\right)e^{-x} \quad - (16)$$

In order to apply this to eq (1), consider its reduced equation:

$$\frac{d^2u}{d\theta^2} + u - \frac{1}{\alpha} = 0 \quad - (17)$$

which is the Newtonian equation of motion. Two independent solutions of eq. (17) are:

$$u_1 = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad - (18)$$

and

$$u_2 = \frac{1}{\alpha} \quad - (19)$$

The complementary function is:

$$u_c = A_1u_1 + A_2u_2 \quad - (20)$$

4) For simplicity chose:

$$A_1 = A_2 = 1 \quad -(21)$$

so

$$u_c = \frac{1}{\alpha} + \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) \quad -(22)$$

Now write eq. (1) as:

$$\frac{d^2u}{d\theta^2} + u - \frac{1}{\alpha} = \delta f^2(\theta) \quad -(23)$$

where

$$u = f(\theta). \quad -(24)$$

The function $f(\theta)$ is to be determined. The general solution of eq (23) is:

$$u = \left(1+v_1(\theta)\right)u_1 + \left(1+v_2(\theta)\right)u_2 \quad -(25)$$

Here:

$$v_1(\theta) = -\int \frac{\delta u_2 f^2(\theta)\, d\theta}{u_1u_2' - u_1'u_2} \quad -(26)$$

$$v_2(\theta) = \delta \int \frac{u_1 f^2(\theta)\, d\theta}{u_1u_2' - u_1'u_2} \quad -(27)$$

From eqns. (18) and (19):

$$u_1' = -\frac{\epsilon}{\alpha}\sin\theta, \quad u_2' = 0. \quad -(28)$$

The Wronskian is therefore:

$$-u_1'u_2 = \frac{\epsilon}{\alpha^2}\sin\theta \quad -(29)$$

5) Therefore:

$$v_1(\theta) = -\frac{\alpha^2\delta}{\epsilon}\int \frac{u_2 f^2(\theta)}{\sin\theta}\,d\theta = -\frac{\alpha\delta}{\epsilon}\int \frac{f^2(\theta)}{\sin\theta}\,d\theta \quad -(21)$$

$$v_2(\theta) = \frac{\alpha^2\delta}{\epsilon}\int \frac{u_1 f^2(\theta)}{\sin\theta}\,d\theta = \frac{\alpha\delta}{\epsilon}\int \frac{(1+\epsilon\cos\theta)}{\sin\theta} f^2(\theta)\,d\theta \quad -(22)$$

So the solution of the EGR equation (1) is:

$$u = \frac{1}{\alpha}\left((1+v_1(\theta))(1+\epsilon\cos\theta) + (1+v_2(\theta))\right) \quad -(23)$$

From eqs. (22) and (23):

$$u = u_c + \frac{1}{\alpha}\left(\epsilon v_1(\theta)\cos\theta + v_2(\theta)\right) \quad -(24)$$

Here:

$$u = u_c + \frac{\delta}{\epsilon}\int \frac{(1+\epsilon\cos\theta)}{\sin\theta} f^2(\theta)\,d\theta - \delta\int \frac{f^2(\theta)}{\sin\theta}\,d\theta \quad -(25)$$

where

$$\delta = \frac{3GM}{c^2} \quad -(26)$$

so the units of $\delta$ are metres. Eq. (25) is therefore dimensionally correct. It can see

6)

written as:

$$u = \left[\frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) + \frac{3GM}{c^2\epsilon}\int\frac{f^2(\theta)}{\sin\theta}\left(1+\epsilon\cos\theta\right)d\theta\right]$$
$$+\left[\frac{1}{\alpha} - \frac{3GM}{c^2}\int\frac{f^2(\theta)}{\sin\theta}d\theta\right] \qquad -(27)$$

The first bracket represents the perturbation of the Newtonian orbit, and the second bracket represents the perturbation of the constant solution $1/\alpha$. For the purposes of astronomy we are interested in the perturbation of the orbit. So the relevant solution is:

$$\boxed{u = \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) + \frac{3GM}{c^2\epsilon}\int f^2(\theta)\left(\frac{1+\epsilon\cos\theta}{\sin\theta}\right)d\theta \qquad -(28)}$$

where

$$u = f(\theta) \qquad -(29)$$

It is seen that eq. (28) is rather a precessing ellipse:

$$u = \frac{1}{\alpha}\left(1+\epsilon\cos(x\theta)\right) \qquad -(30)$$

7) In the solar system and even in systems such as binary pulsars, the precession of the perihelion is very small. For example in the solar system:

$$x - 1 \sim 10^{-6} \quad -(31)$$

So to a very good approximation:

$$f^2(\theta) = \frac{1}{\alpha^2}\left(1+\epsilon\cos\theta\right)^2 \quad -(32)$$

and:

$$u \sim \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) + \frac{3GM}{c^2\epsilon\alpha^2}\int\frac{(1+\epsilon\cos\theta)^3}{\sin\theta}\,d\theta \quad -(33)$$

and this is never a precessing ellipse (30)

The rigorously correct equation is:

$$f(\theta) = \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) + \frac{3GM}{c^2\epsilon}\int f^2(\theta)\left(\frac{1+\epsilon\cos\theta}{\sin\theta}\right)d\theta \quad -(34)$$

Differentiating:

$$\frac{df}{d\theta} = -\frac{\epsilon}{\alpha}\sin\theta + \frac{3GM}{c^2\epsilon}\left(\frac{1+\epsilon\cos\theta}{\sin\theta}\right)f^2(\theta) \quad -(35)$$

so the second order equation (1) has been reduced to

8) the first order eq. (35).

<u>Estimate of Correction</u>:

$$G = 6.67384(80) \times 10^{-11}\ \mathrm{N m^2 kg^{-2}}$$

$$M = 1.9891 \times 10^{30}\ \mathrm{kg}.$$

$$c = 2.998 \times 10^{8}\ \mathrm{m s^{-1}}$$

$$\alpha \doteq r = 1.496 \times 10^{12}\ \mathrm{m}$$

$$\epsilon \doteq 1$$

where $r$ is the earth sun distance and $\epsilon$ the eccentricity of the earth's orbit. So:

$$\frac{3GM}{c^2 \alpha^2 \epsilon} \sim \frac{10^{-11} \times 10^{30}}{10^{16} \times 10^{24}} \quad -(36).$$

$$= \text{order } 10^{-21}$$

Therefore the second term in eq. (33) is a very tiny correction to the Newtonian solution, <u>and eq. (33) cannot be a precessing ellipse for any $\epsilon$ and $\alpha$. under any circumstances</u>

<u>Conclusion</u>

It is simple to show that the claims of Einsteinian general relativity are wildly incorrect.