223(2): The Basic Relativistic and Classical Hamiltonians.

In the constrained Minkowski method the starting metric is a plane $d\tau^2 = 0$ is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

For a free particle the relativistic hamiltonian and lagrangian are:

$$\mathcal{L} = H = \frac{1}{2} m c^2 = \frac{1}{2} m c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{m}{2} r^2 \left(\frac{d\theta}{d\tau}\right)^2 \quad -(2)$$

The relativistic Euler Lagrange equations give:

$$E = \gamma m c^2 = m c^2 \left(\frac{dt}{d\tau}\right) \quad -(3)$$

$$L = m r^2 \frac{d\theta}{d\tau} = \gamma m r^2 \frac{d\theta}{dt} \quad -(4)$$

Eq. (2) is:

$$\mathcal{L} = H = \frac{1}{2} m c^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - p^2\right) \quad -(5)$$

Here:

$$\underline{p} = m \frac{d\underline{r}}{d\tau} \quad -(6)$$

is the relativistic momentum. Therefore:

$$p^2 = m^2 \frac{d\underline{r}}{d\tau} \cdot \frac{d\underline{r}}{d\tau} \quad -(7)$$

In plane polar coordinates:

2)
$$\frac{d\mathbf{r}}{d\tau} \cdot \frac{d\mathbf{r}}{d\tau} = \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2 \quad -(8)$$

From eq. (6):
$$\underline{p} = \gamma m \underline{v} \quad -(9)$$

where
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(10)$$

It follows (see UFT papers and Marion and Thornton) that:
$$E^2 = c^2 p^2 + m^2 c^4 \quad -(11)$$

so
$$p^2 = \frac{E^2 - m^2c^4}{c^2} \quad -(12)$$

Therefore:
$$\frac{p^2}{2m} = \frac{m}{2}\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) = \frac{E^2 - m^2c^4}{2mc^2} \quad -(13)$$

In the classical limit:
$$v \ll c, \quad -(14)$$

the kinetic energy is:
$$T = \frac{1}{2} m v^2 = \frac{p^2}{2m} \quad -(15)$$

and for a free particle in the classical limit:

3)

$$\mathcal{L} = E = T. \quad -(16)$$

In special relativity however the relativistic kinetic energy of a free particle is:

$$T = (\gamma - 1)mc^2, \quad -(17)$$

which is obtained from:

$$T = W = \int \underline{F} \cdot \underline{v}\, dt \quad -(18)$$

From eqs. (11) and (17):

$$E = \gamma mc^2 = T + mc^2, \quad -(19)$$

So for a free particle in special relativity E is the relativistic kinetic energy added to the rest energy:

$$E_0 = mc^2. \quad -(20)$$

The concept of rest energy does not exist in classical dynamics.

Therefore in eq. (13):

$$\frac{p^2}{2m} = \frac{1}{2} mc^2 (\gamma^2 - 1). \quad -(21)$$

From eqs. (17) and (21)

$$\frac{p^2}{2m} = \frac{1}{2} mc^2 (\gamma - 1)(\gamma + 1) = \frac{1}{2} T(\gamma + 1) \quad -(22)$$

4) So in special relativity:

$$\boxed{T = \frac{p^2}{(\gamma+1)m}} \quad -(23)$$

where:

$$p^2 = m^2\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(24)$$

$$= m^2 \frac{d\underline{r}}{d\tau} \cdot \frac{d\underline{r}}{d\tau}$$

In special relativity, the constant quantity is:

$$mc^2 = E - T, \quad -(25)$$

so neither E nor T is constant in special relativity. The classical lagrangian is obtained from:

$$\mathcal{L} = E - mc^2 \quad -(26)$$

$$= \frac{p^2}{m(\gamma+1)}$$

in the limit:

$$v \ll c. \quad -(27)$$

In the limit (27):

5) $$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \rightarrow 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots \quad -(28)$$

so in eq. (23):

$$T \rightarrow \frac{p^2}{\left(2 + \frac{1}{2}\frac{v^2}{c^2}\right)m} \quad -(29)$$

$$\sim \frac{p^2}{2m}.$$

Also:

$$\frac{p^2}{2m} = \frac{1}{2}mc^2\left(\gamma^2 - 1\right)$$

$$\rightarrow \frac{1}{2}mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)$$

$$\rightarrow \frac{1}{2}mc^2\left(1 + \frac{v^2}{c^2} - 1\right) \quad -(30)$$

$$= \frac{1}{2}mv^2,$$

and

$$T = (\gamma - 1)mc^2$$

$$\rightarrow \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2$$

$$= \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right)mc^2 \quad -(31)$$

$$= \frac{1}{2}mv^2.$$