223(5): Relativistic Orbital Force

The relativistic orbital lagrangian is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U(r) \quad -(1)$$

$$= -mc^2\left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right)^{1/2} - U(r)$$

using $v^2 = \dot{r}^2 + r^2\dot{\theta}^2. \quad -(2)$

The two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(3)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}. \quad -(4)$$

Define:

$$f(\dot{\theta}) = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right)^{1/2} \quad -(5)$$

then

$$\frac{\partial \mathcal{L}}{\partial f} = -\frac{1}{2}mc^2 f(\dot{\theta})^{-1/2} = -\frac{1}{2}\gamma mc^2 \quad -(6)$$

We have:

$$\frac{\partial f}{\partial \dot{\theta}} = -\frac{2r^2}{c^2}\dot{\theta} \quad -(7)$$

So

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \dot{\theta}} = \gamma m r^2\dot{\theta} \quad -(7)$$

2) The relativistic angular momentum is:

$$\boxed{L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \gamma m r^2 \dot{\theta}} \quad -(8)$$

This is the same result as obtained from the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(9)$$

for a free particle.

Similarly:

$$\frac{\partial f}{\partial \dot{r}} = -\frac{2\dot{r}}{c^2}, \quad \frac{\partial f}{\partial r} = -\frac{2r\dot{\theta}^2}{c^2} \quad -(10)$$

so

$$\frac{\partial \mathcal{L}}{\partial r} = \gamma m r \dot{\theta}^2 - \frac{\partial U}{\partial r} \quad -(11)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \gamma m \dot{r} \quad -(12)$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \gamma m \ddot{r} \quad -(13)$$

So eq. (3) gives:

$$\gamma m \left(\ddot{r} - r\dot{\theta}^2\right) = F(r) \quad -(14)$$

$$L = \gamma m r^2 \dot{\theta} \quad -(15)$$

These are the same as the non-relativistic

3) equivalents, eqs. (7.18) and (7.10) of Marion and Thornton, 3rd edition, except that:

$$m \rightarrow \gamma m. \quad -(16)$$

It follows that the relativistic force equation is:

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma m r^2}{L^2} F(r)} \quad -(17)$$

The observed orbit in the solar system is:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos(x\theta)\right) \quad -(18)$$

From eqs. (17) and (18):

$$F(r) = -\frac{x^2 L^2}{\gamma m r^2 \alpha} - \frac{(1-x^2) L^2}{\gamma m r^3} \quad -(19)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(20)$$

Thus:

$$\boxed{F(r) = \left(1 - \frac{v^2}{c^2}\right)^{1/2} F(\text{classical})} \quad -(21)$$

where

$$F(\text{classical}) = -\frac{x^2 L^2}{m r^2 \alpha} - \frac{(1-x^2) L^2}{m r^3} \quad -(22)$$