223(6): Summary of Orbital Theory in Special Relativity.

All quantities in special relativity are denoted by $*$. The linear momentum is:

$$p^* = \gamma m v = \frac{\partial \mathcal{L}^*}{\partial v} \quad -(1)$$

where:

$$\mathcal{L}^* = T^* - U(r), \quad -(2)$$

$$T^* = -\frac{mc^2}{\gamma}, \quad -(3)$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad -(4)$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2. \quad -(5)$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}^*}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{\theta}} \quad -(6)$$

and

$$\frac{\partial \mathcal{L}^*}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{r}}. \quad -(7)$$

The r dependent force is:

$$F(r) = -\frac{\partial U}{\partial r}. \quad -(8)$$

These equations give the relativistic orbital equation:

2)

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma m}{L^{*2}} r^2 F(r) \quad -(9)$$

where the relativistic angular momentum is:

$$L^* = \gamma m r^2 \dot{\theta}. \quad -(10)$$

Therefore:

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{1}{\gamma}\left(\frac{m r^2 F(r)}{L^2}\right)} \quad -(11)$$

The non-relativistic result is:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m r^2 F(r)}{L^2} \quad -(12)$$

where

$$L = m r^2 \dot{\theta}. \quad -(13)$$

Eq. (11) is a powerful and elegant result valid for all observed orbits.

The static ellipse is:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos\theta\right) \quad -(14)$$

So:

3)

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}, \quad -(15)$$

and

$$\frac{1}{\alpha} = -\frac{1}{\gamma}\left(\frac{mr^2F(r)}{L^2}\right). \quad -(16)$$

In order for special relativity to produce a static ellipse:

$$F^*(r) = -\gamma\left(\frac{L^2}{m\alpha r^3}\right) = \gamma F(r) \quad -(17)$$

Also in special relativity:

$$F^* = \frac{dp^*}{dt} \quad -(18)$$

$$= \gamma \frac{dp}{dt}$$

In eq. (17):

$$\alpha = \frac{L^2}{m^2 MG} \quad -(19)$$

if

$$F = -\frac{mMG}{r^2} \quad -(20)$$

so

$$\boxed{F^* = \gamma F = -\frac{\gamma mMG}{r^2} = \gamma \frac{dp}{dt}} \quad -(21)$$

4) For a precessing ellipse:

$$F^* = \gamma F,$$

$$F = \frac{dp}{dt} = -\frac{x^2 L^2}{m r^2 \alpha} - \frac{(1-x^2)L^2}{m r^3}$$

$$= -x^2 \frac{mMG}{r^2} + \frac{(x^2-1)L^2}{m r^3} \quad -(22)$$

and:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos(x\theta)\right) \quad -(23)$$

For any orbit:

$$\boxed{F^* = \gamma F} \quad -(24)$$

The infinitesimal line element is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(25)$$

where

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2 \quad -(26)$$

$$= dr^2 + r^2 d\theta^2 .$$

So:

$$c^2 d\tau^2 = (c^2 - v^2) dt^2 \quad -(27)$$

and

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt = \gamma dt. \quad -(28)$$

5) The orbit is described by the force law (24). The metric $g_{\mu\nu}$ is defined by:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad -(29)$$

The observed orbit produces the constraint:

$$g = \frac{dr}{d\theta} \quad -(30)$$

so:

$$ds^2 = c^2 dt^2 - \left(1 + \frac{1}{f}\right) r^2 d\theta^2 \quad -(31)$$

where:

$$f = \left(\frac{r}{g}\right)^2 = r^2 \left(\frac{d\theta}{dr}\right)^2. \quad -(32)$$

Therefore, the relativistic time dilation is:

$$\frac{d\tau}{dt} = \left(1 - \frac{1}{c^2}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)\left(\frac{d\theta}{dt}\right)^2\right)^{1/2} \quad -(33).$$

As in UFT208:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -\left(1 + \frac{1}{f}\right) \end{bmatrix}, \quad -(34)$$

$$g_{00} = 1, \quad g_{22} = -\left(1 + \frac{1}{f}\right) \quad -(35)$$

6) The orbit is caused by the Cartan torsion:

$$T^{2}{}_{02} = -T^{2}{}_{20} = \frac{1}{c}\left(1+\frac{1}{f}\right)^{-1}\frac{\partial}{\partial t}\left(1+\frac{1}{f}\right) \quad -(36)$$

The Cartan / Evans identity gives

$$\frac{d}{d\theta}\left(\omega \frac{df}{d\theta}\right) = 0 \quad -(37)$$

and

$$\frac{d\omega}{d\theta} = -F(\theta)\,\omega \quad -(37)$$

where

$$\omega = \frac{d\theta}{dt}. \quad -(38)$$

The fundamental cause of orbits is the Cartan torsion and curvature. The relativistic orbital force equation is eq. (11). The relativistic lagrangian is eq. (2), and the relativistic Euler Lagrange equations are eqs. (6) and (7), which are equivalent to eq. (11). Eq. (21) is the relativistic equivalence principle.