223(4): The Lagrangian in Special Relativity.

In order to construct a fully consistent theory of cosmology a well defined relativistic lagrangian is needed. For the contravariant Minkowski metric the lagrangian must be the one developed for special relativity. The method given here follows Marion and Thornton, 3rd ed., pp 538 ff.

For a single non-relativistic particle moving in a velocity independent potential the Cartesian components of momentum are:

$$p_i = \frac{\partial \mathcal{L}}{\partial v_i} \quad -(1)$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

the relativistic momentum is:

$$p_i = \gamma m v_i \quad -(2)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

The relativistic lagrangian is constructed to give eq. (2), and for each i:

$$\frac{\partial \mathcal{L}}{\partial v} = \gamma m v \quad -(4)$$

The velocity independent part of the lagrangian is the potential energy U, and this is the same as in classical dynamics.

a)

Therefore the relativistic Lagrangian is:

$$\mathcal{L} = T^* - U \quad -(5)$$

where

$$T^* = T^*(v); \quad U = U(x) \quad -(6)$$

Here:

$$\frac{\partial T^*}{\partial v} = \gamma m v, \quad -(7)$$

and one possibility is:

$$T^* = -mc^2/\gamma$$
$$= -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} \quad -(8)$$

Therefore:

$$\boxed{\mathcal{L} = -\frac{mc^2}{\gamma} - U} \quad -(9)$$

The Hamiltonian is calculated from:

$$H = \sum_i v_i p_i - \mathcal{L}. \quad -(10)$$

From eqs. (2) and (9), for one particle:

$$H = \frac{p^2c^2}{\gamma mc^2} + \frac{mc^2}{\gamma} + U$$
$$= \frac{1}{\gamma mc^2}\left(p^2c^2 + m^2c^4\right) + U \quad -(11)$$

So:

$$\boxed{H = \frac{E^2}{\gamma m c^2} + V} \quad -(12)$$

However:

$$E = \gamma m c^2 \quad -(13)$$

So:

$$\boxed{\begin{aligned} H &= E + U \\ &= T + E_0 + U \end{aligned}} \quad -(14)$$

where

$$E_0 = mc^2. \quad -(15)$$

Note carefully that the relativistic kinetic energy is

$$T = (\gamma - 1) mc^2 \quad -(16)$$

and that the lagrangian of special relativity is no longer defined by:

$$\mathcal{L} = T - U. \quad -(17)$$

For relativistic orbital theory:

$$\boxed{\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMGx^2}{r} + \frac{(x^2-1)L^2}{2mr^2}} \quad -(18)$$

4) The classical limit of eq. (18) is defined by:

$$v << c \quad - (19)$$

so:

$$\mathcal{L} \to -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} + \frac{mMGx^2}{r} + \frac{(x^2-1)L^2}{2mr^2} \quad - (20)$$

$$= -mc^2\left(1-\frac{1}{2}\frac{v^2}{c^2}+\dots\right) + \frac{mMGx^2}{r} + \frac{(x^2-1)L^2}{2mr^2}$$

$$= \frac{1}{2}mv^2 + \frac{mMGx^2}{r} + \frac{(x^2-1)L^2}{2mr^2}$$

which is the classical lagrangian of x theory, QED.