223(7): Cross Check on the Evaluation of Relativistic Force.

Consider the velocity and acceleration in cylindrical polar coordinates:

$$\underline{v} = \dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\underline{e}_r\right) \quad -(1)$$

$$= \dot{r}\underline{e}_r + r\dot{\underline{e}}_r ,$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta . \quad -(2)$$

The relativistic angular momentum is:

$$L = \gamma m r^2 \dot{\theta} \quad -(3)$$

and the orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} . \quad -(4)$$

Therefore

$$\frac{dr}{d\theta} = \frac{\epsilon}{\alpha} r^2 \sin\theta \quad -(5)$$

and

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{\epsilon L}{\gamma m \alpha}\sin\theta . \quad -(6)$$

So:

$$\dot{r} = \frac{1}{\gamma}\left(\frac{L\epsilon}{m\alpha}\right)\sin\theta , \quad \dot{\theta} = \frac{L}{\gamma m r^2} \quad -(7)$$

It follows that

$$\ddot{r} = \frac{1}{\gamma}\left(\frac{L\epsilon}{m\alpha}\right)\frac{d}{dt}\sin\theta \quad -(8)$$

$$= \frac{1}{\gamma}\left(\frac{L\epsilon}{m\alpha}\right)\cos\theta\,\frac{d\theta}{dt}$$

2)

So
$$\ddot{r} = \left(\frac{L}{\gamma m r}\right)^2 \frac{\epsilon \cos\theta}{\alpha} \quad -(9)$$

Also:
$$\ddot{\theta} = -\frac{2L}{\gamma m r^3}\frac{1}{\gamma}\left(\frac{L\epsilon}{m\alpha}\right)\sin\theta$$
$$= -2\left(\frac{L}{\gamma m}\right)^2 \frac{\epsilon}{\alpha}\frac{1}{r^3}\sin\theta \quad -(10)$$

So:
$$\ddot{r} - r\dot{\theta}^2 = \left(\frac{L}{\gamma m r}\right)^2\left(\frac{\epsilon}{\alpha}\cos\theta - \frac{1}{r}\right) \quad -(11)$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(12)$$

Therefore:
$$\underline{a} = \left(\frac{L}{\gamma m r}\right)^2\left(\frac{\epsilon}{\alpha}\cos\theta - \frac{1}{r}\right)\underline{e}_r \quad -(13)$$

where
$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(14)$$

It follows that:
$$\boxed{\underline{a} = -\frac{1}{\alpha}\left(\frac{L}{\gamma m r}\right)^2 \underline{e}_r} \quad -(15)$$

This is the relativistic acceleration due to eq. (4).

3) The relativistic force is defined by:

$$\underline{F} = m \frac{d}{dt}(\gamma \underline{v}) \quad - (16)$$

So

$$F = \frac{d}{dt}(\gamma m v) = m\left(\frac{d\gamma}{dt} v + \gamma \frac{dv}{dt}\right) \quad - (17)$$

where:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad - (18)$$

$$= \left(\frac{d}{dv}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right)\frac{dv}{dt}$$

$$= \frac{v}{c^2}\frac{dv}{dt}\gamma^3$$

So

$$F = m\left(\frac{v^2}{c^2}\frac{dv}{dt}\gamma^3 + \gamma\frac{dv}{dt}\right)$$

$$= m\gamma\frac{dv}{dt}\left(\gamma^2\frac{v^2}{c^2} + 1\right)$$

$$= m\gamma\frac{dv}{dt}\left(\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1} + 1\right)$$

$$\boxed{F = m\gamma^3\frac{dv}{dt} = m\gamma^3 a} \quad - (19)$$

4) From eqs. (15) and (19):

$$\underline{F} = -\frac{\gamma}{\alpha}\left(\frac{L}{mr}\right)^2 \underline{e}_r \quad -(20)$$

and this is the same as eq. (17) of note 223(6).

<u>Conclusion</u>

The relativistic force needed to describe the ellipse (4) is

$$\boxed{F^* = \gamma F} \quad -(21)$$

where F is the non-relativistic force, i.e.

$$F^* = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} F \quad -(22)$$

or

$$\boxed{\begin{aligned} F^* &= \left(1 - \frac{v^2}{c^2}\right)^{-3/2} m \frac{dv}{dt} \\ &= \left(1 - \frac{v^2}{c^2}\right)^{-3/2} m \frac{d^2 r}{dt^2} \end{aligned}}$$