221(4): Stokes Theorem for a Circle

In this case:

$$\oint \underline{R} \cdot d\underline{R} = \int \underline{\nabla} \times \underline{R} \cdot \underline{n}\, dA \quad -(1)$$

$$= 0$$

so the circulation vector is zero:

$$\underline{S} = \underline{\nabla} \times \underline{R}. \quad -(2)$$

This is proven as follows:

$$\underline{R} = X\underline{i} + Y\underline{j} \quad -(3)$$

$$d\underline{R} = \underline{i}\, dX + \underline{j}\, dY. \quad -(4)$$

Therefore: $$\oint \underline{R} \cdot d\underline{R} = \int_0^{2\pi} (X\,dX + Y\,dY). \quad -(5)$$

In order to evaluate this:

$$X = R\cos\theta, \quad Y = R\sin\theta, \quad -(6)$$

$$\frac{dX}{d\theta} = -R\sin\theta, \quad \frac{dY}{d\theta} = R\cos\theta, \quad -(7)$$

$$dX = -R\sin\theta\, d\theta, \quad dY = R\cos\theta\, d\theta. \quad -(8)$$

So: $$\int_0^{2\pi} (X\,dX + Y\,dY)$$

$$= \int_0^{2\pi} R\cos\theta(-R\sin\theta)\, d\theta + R\sin\theta(R\cos\theta)\, d\theta \quad -(9)$$

$$= 0$$

and the circulation or curl is evaluated as follows:

2)

$$\nabla \times \underline{R} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} \quad -(10)$$

$$= \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \underline{k}.$$

$$= 0$$

In cylindrical polar coordinates:

$$\underline{R} = R\,\underline{e}_r \quad -(11)$$

where

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(12)$$

and

$$\nabla \times \underline{R} = -\frac{1}{R}\frac{\partial R}{\partial \theta}\,\underline{k}. \quad -(13)$$

The circle is defined by:

$$R = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(14)$$

with

$$\epsilon = 0. \quad -(15)$$

so

$$R = \alpha = \text{constant} \quad -(16)$$

and

$$\frac{\partial R}{\partial \theta} = 0, \quad -(17)$$

so

$$\nabla \times \underline{R} = \underline{0}. \quad -(18)$$
