221(2): Orbital Concepts from Stokes' Theorem

In the Newtonian limit:

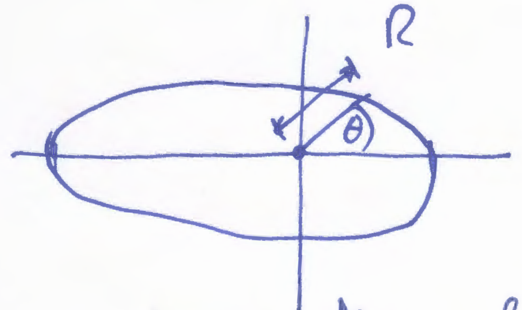
$$R = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(1)$$

corresponds to:

$$\underline{R} = (a\epsilon + X)\,\underline{i} + Y\,\underline{j} \quad -(2)$$

with:

$$X = R\cos\theta, \quad Y = R\sin\theta \quad -(3)$$



This is an ellipse centred at one focus. Therefore:

$$\frac{\partial \underline{R}}{\partial R} = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(4)$$

$$\frac{\partial \underline{R}}{\partial \theta} = -R\sin\theta\,\underline{i} + R\cos\theta\,\underline{j} \quad -(5)$$

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(6)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(7)$$

$$\underline{i} = \underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta \quad -(8)$$

$$\underline{j} = \underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta \quad -(9)$$

Therefore:

2)
$$\underline{R} = (a\epsilon + R\cos\theta)(\underline{e}_r \cos\theta - \underline{e}_\theta \sin\theta) + R\sin\theta\,(\underline{e}_r \sin\theta + \underline{e}_\theta \cos\theta)$$

$$\boxed{\underline{R} = (a\epsilon\cos\theta + R)\,\underline{e}_r - a\epsilon\sin\theta\,\underline{e}_\theta} \quad -(10)$$

The Stokes Theorem is:

$$\oint \underline{R}\cdot d\underline{R} = \int \underline{\nabla}\times\underline{R}\cdot\underline{n}\,dA \quad -(11)$$

where:

$$d\underline{R} = \underline{e}_r\,dR + \underline{e}_\theta\,R\,d\theta \quad -(12)$$

Here:

$$dA = \frac{1}{2}R^2\,d\theta. \quad -(13)$$

We introduce the concept of orbital circulation or swirl:

$$\underline{S} = \underline{\nabla}\times\underline{R} \quad -(14)$$

This is defined by:

$$\underline{\nabla}\times\underline{R} = \frac{1}{R}\left(\frac{\partial(RR_\theta)}{\partial R} - \frac{\partial R_r}{\partial\theta}\right)\underline{k} \quad -(15)$$

where:

$$R_r = a\epsilon\cos\theta + R \quad -(16)$$

$$R_\theta = -a\epsilon\sin\theta \quad -(17)$$

i.e.

3)

$$\underline{R} = R_r \underline{e}_r + R_\theta \underline{e}_\theta \quad - (18)$$

Using the Leibnitz Theorem:

$$\underline{\nabla} \times \underline{R} = \frac{1}{R}\left(R_\theta + R\frac{\partial R_\theta}{\partial R} - \frac{\partial R_r}{\partial \theta}\right)\underline{k} . \quad - (19)$$

However:

$$\frac{\partial R_\theta}{\partial R} = 0 \quad - (20)$$

So:

$$\boxed{\underline{S} = \underline{\nabla} \times \underline{R} = \frac{1}{R}\left(R_\theta - \frac{\partial R_r}{\partial \theta}\right)\underline{k}} \quad - (20)$$

i.e.

$$\underline{S} = \underline{\nabla} \times \underline{R} = \frac{1}{R}\left(-a\epsilon \sin\theta + a\epsilon \sin\theta - \frac{\partial R}{\partial \theta}\right)\underline{k}$$

$$\boxed{\underline{S} = -\frac{1}{R}\frac{\partial R}{\partial \theta}\underline{k}} \quad - (21)$$

This is the same result as note 221(1)

Assuming that:

$$\underline{k} \cdot \underline{n} = 1 \quad - (22)$$

then:

$$\boxed{\oint \underline{R} \cdot d\underline{R} = -\frac{1}{2}\int R^2 \frac{\partial R}{\partial \theta}\, d\theta} \quad - (23)$$

From eq. (1):

4)

$$\underline{S} = -\frac{\epsilon}{\alpha} R \sin\theta \, \underline{k}$$

i.e.

$$\boxed{\underline{S} = -\frac{\epsilon \sin\theta}{1+\epsilon\cos\theta} \underline{k}} \quad -(24)$$

and

$$\boxed{\oint \underline{R} \cdot \underline{dR} = -\frac{\alpha^2 \epsilon}{2} \int \frac{\sin\theta \, d\theta}{(1+\epsilon\cos\theta)^3}} \quad -(25)$$

For the precessing orbit:

$$\boxed{\underline{S} = -\frac{\epsilon x \sin(x\theta)}{1+\epsilon\cos(x\theta)}} \quad -(26)$$

$$\boxed{\oint \underline{R} \cdot \underline{dR} = -\frac{\alpha^2 \epsilon x}{2} \int \frac{\sin(x\theta) \, d\theta}{(1+\epsilon\cos(x\theta))^3}} \quad -(27)$$

Plots of eqns. (26) and (27) can now be made.