221(1): Complete Analytical Solution of the Three-Particle Problem

As in previous notes the lagrangian for the three particle problem is:

$$\mathcal{L} = \frac{1}{2}\left(m_1 |\dot{\underline{r}}_1|^2 + m_2 |\dot{\underline{r}}_2|^2 + m_3 |\dot{\underline{r}}_3|^2\right) - \frac{m_1 m_2 G}{R_1} - \frac{m_1 m_3 G}{R_2} - \frac{m_2 m_3 G}{R_3} \quad -(1)$$

where

$$R_1 = |\underline{r}_1 - \underline{r}_2| \quad -(2)$$

$$R_2 = |\underline{r}_1 - \underline{r}_3| \quad -(3)$$

$$R_3 = |\underline{r}_2 - \underline{r}_3| \quad -(4)$$

Now use:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = 0 \quad -(5)$$

and

$$\underline{R}_1 = \underline{r}_1 - \underline{r}_2 . \quad -(6)$$

Therefore:

$$m_1 |\dot{\underline{r}}_1|^2 + m_2 |\dot{\underline{r}}_2|^2 = \mu_1 |\dot{\underline{R}}_1|^2 \quad -(7)$$

where

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2} \quad -(8)$$

The lagrangian (1) therefore becomes:

$$\mathcal{L} = \frac{1}{2}\left(\mu_1 |\dot{\underline{R}}_1|^2 + m_3 |\dot{\underline{r}}_3|^2\right) - \frac{m_1 m_2 G}{R_1} - \frac{m_1 m_3 G}{R_2} - \frac{m_2 m_3 G}{R_3} \quad -(9)$$

where:

$$|\dot{\underline{R}}_1|^2 = \dot{R}_1^2 + R_1^2 \dot{\theta}_1^2 \quad -(10)$$

Now consider the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial R_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}_1} \quad -(11)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \quad -(12)$$

These can be written as:

$$\frac{d^2}{d\theta_1^2}\left(\frac{1}{R_1}\right) + \frac{1}{R_1} = -\frac{\mu_1 R_1^2}{L_1^2} F_1(R_1) \quad -(13)$$

where

$$L_1 = \mu_1 R_1^2 \frac{d\theta_1}{dt} \quad -(14)$$

$$F_1 = -\frac{\partial U_1}{\partial R_1} \quad -(15)$$

and

$$U_1 = -\frac{m_1 m_2 G}{R_1} \quad -(16)$$

The solution of eq. (13) is:

$$R_1 = \frac{\alpha_1}{1 + \epsilon_1 \cos\theta_1} \quad -(17)$$

where

$$L_1^2 = \alpha_1 \mu_1 m_1 m_2 G, \quad -(18)$$

and

$$\epsilon_1 = \left(1 + \frac{2E_1 L_1^2}{\mu_1 m_1^2 m_2^2 G^2}\right)^{1/2} \quad -(19)$$

Eq. (17) is an ellipse with major and minor semi-axes:

$$a_1 = \frac{\alpha_1}{1-\epsilon_1^2} = \frac{m_1 m_2 G}{2|E_1|} \quad -(20)$$

$$b_1 = \frac{\alpha_1}{(1-\epsilon_1^2)^{1/2}} = \frac{L_1}{(2\mu_1|E_1|)^{1/2}} \quad -(21)$$

So:

$$\boxed{\begin{aligned} \epsilon_1^2 &= 1 - \left(\frac{b_1}{a_1}\right)^2 \quad -(22) \\ \alpha_1 &= a_1(1-\epsilon_1^2) \quad -(23) \\ &= \frac{b_1^2}{a_1} \end{aligned}}$$

4) Therefore α_1 and ϵ_1 can be measured in astronomy. The actually measured quantities are a_1 and b_1. They are related to the distance of closest approach, r_{min}, and furthest distance away, r_{max}, by:

$$r_{min} = a(1-\epsilon) \quad -(24)$$

$$r_{max} = a(1+\epsilon) \quad -(25)$$

Similarly:

$$R_2 = \frac{\alpha_2}{1+\epsilon_2 \cos\theta_2}, \quad -(26)$$

$$L_2 = \mu_2 R_2^2 \frac{d\theta_2}{dt}, \quad -(27)$$

$$\mu_2 = \frac{m_1 m_3}{m_1 + m_3} \quad -(28)$$

$$\epsilon_2^2 = 1 - \left(\frac{b_2}{a_2}\right)^2, \quad \alpha_2 = \frac{b_2^2}{a_2}. \quad -(29)$$

Thirdly:

$$R_3 = \frac{\alpha_3}{1+\epsilon_3 \cos\theta_3}, \quad -(30)$$

$$L_3 = \mu_3 R_3^2 \frac{d\theta_3}{dt} \quad -(31)$$

$$\mu_3 = \frac{m_2 m_3}{m_2 + m_3}, \quad -(32)$$

$$\epsilon_3^2 = 1 - \left(\frac{b_3}{a_3}\right)^2, \quad \alpha_3 = \frac{b_3^2}{a_3}. \quad -(33)$$

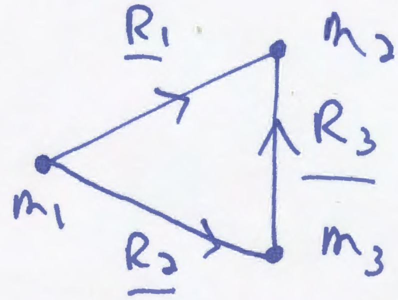

Fig (1)

The three ellipses are related by:

$$\boxed{\underline{R_1} = \underline{R_2} + \underline{R_3}} \quad -(34)$$

The elapsed times for each ellipse are:

$$t_i = \left(1 - \epsilon_i^2\right)^{3/2} \left(\frac{\tau_i}{2\pi}\right) \int \frac{d\theta}{\left(1 + \epsilon_i \cos\theta\right)^2}$$

$-(35)$

where τ_i are the times for a particle on each ellipse to sweep out 2π radians.

In cylindrical polar coordinates:

$$\underline{R_1} = R_1 \underline{e}_r \quad -(36)$$

6) Similarly:

$$\underline{R_2} = R_2 \, \underline{e}_r \quad -(37)$$

$$\underline{R_3} = R_3 \, \underline{e}_r \quad -(38)$$

From Stokes' Theorem:

$$\oint \underline{R} \cdot d\underline{r} = \int \underline{\nabla} \times \underline{R} \cdot \underline{n} \, dA \quad -(39)$$

where

$$\underline{S} := \underline{\nabla} \times \underline{R} . \quad -(40)$$

In cylindrical polar coordinates, if:

$$\underline{F} = F_r \, \underline{e}_r + F_\theta \, \underline{e}_\theta + F_z \, \underline{k} \quad -(41)$$

then:

$$\underline{\nabla} \times \underline{F} = \left(\frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \underline{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \underline{e}_\theta + \frac{1}{r} \left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \underline{k} . \quad -(42)$$

So:

$$\underline{\nabla} \times \underline{R_i} = -\frac{1}{R_i} \frac{\partial R_i}{\partial \theta_i} \underline{k} \quad -(43)$$

$$i = 1, 2, 3$$

Therefore:

$$\oint \underline{R_i} \cdot d\underline{r} = -\int \frac{1}{R_i} \frac{\partial R_i}{\partial \theta_i} \, dA_i . \quad -(44)$$

$$i = 1, 2, 3$$

In each case:

$$dA_i = \frac{1}{2} R_i^2 \, d\theta_i \quad - (45)$$

$$i = 1, 2, 3$$

Therefore:

$$\oint \underline{R_i} \cdot \underline{dr} = -\frac{1}{2} \int R_i \, dR_i \quad - (46)$$

and it follows that:

$$\boxed{\int R_1 \, dR_1 = \int R_2 \, dR_2 + \int R_3 \, dR_3} \quad - (47)$$

<u>This equation relates to three ellipses.</u>

For each ellipse:

$$R = \frac{\alpha}{1 + \epsilon \cos\theta}, \quad \frac{dR}{d\theta} = \frac{\epsilon}{\alpha} R^2 \sin\theta,$$

$$dR = \frac{\epsilon}{\alpha} R^2 \sin\theta \, d\theta \quad - (48)$$

So:

$$\frac{\epsilon_1}{\alpha_1} \int R_1^3 \sin\theta \, d\theta = \frac{\epsilon_2}{\alpha_2} \int R_2^3 \sin\theta \, d\theta + \frac{\epsilon_3}{\alpha_3} \int R_3^3 \sin\theta \, d\theta$$

$$- (49)$$

8)

i.e.

$$\boxed{\frac{\epsilon_1}{d_1} R_1^3 = \frac{\epsilon_2}{d_2} R_2^3 + \frac{\epsilon_3}{d_3} R_3^3} \quad - (50)$$

These results follows from the fact that:

$$\int R_1^3 \sin\theta_1 \, d\theta_1 = \int R_1^3 \sin\theta \, d\theta \quad - (51)$$

$$\int R_2^3 \sin\theta_2 \, d\theta_2 = \int R_2^3 \sin\theta \, d\theta \quad - (52)$$

$$\int R_3^3 \sin\theta_3 \, d\theta_3 = \int R_3^3 \sin\theta \, d\theta \quad - (53)$$