221 (3): Relative Motion of Centres of Mass

Consider Fig (1):

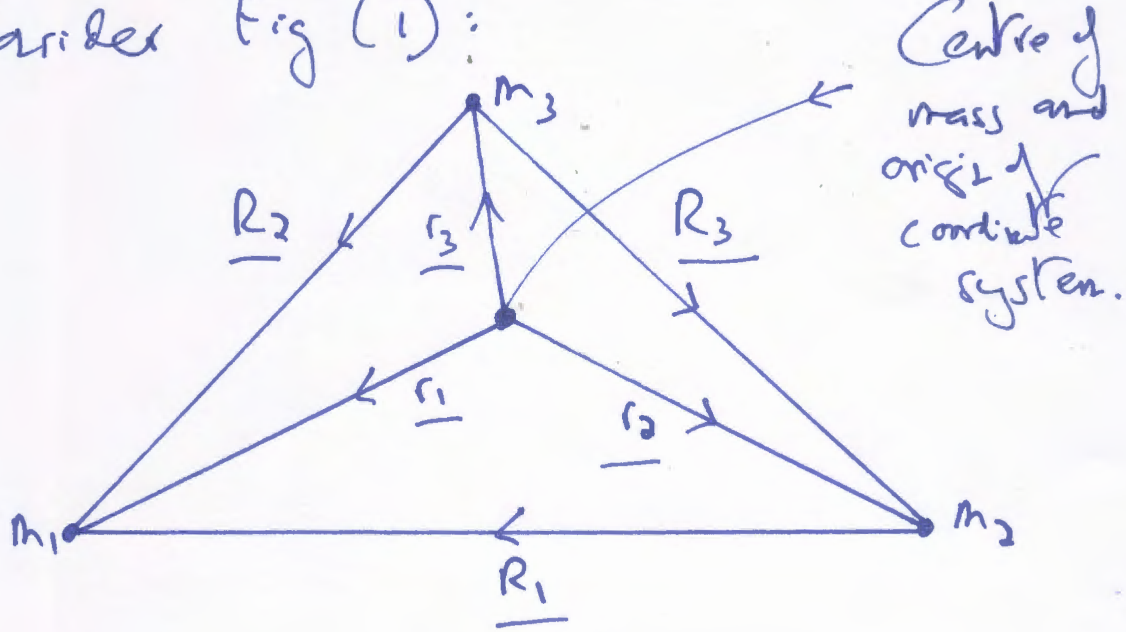

Fig (1)

The centre of mass is at the origin, so it is defined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 + m_3 \underline{r}_3 = \underline{0} \quad (1)$$

Also:

$$\underline{R}_1 = \underline{r}_1 - \underline{r}_2 \quad (2)$$

$$\underline{R}_2 = \underline{r}_1 - \underline{r}_3 \quad (3)$$

$$\underline{R}_3 = \underline{r}_2 - \underline{r}_3 \quad (4)$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(m_1 |\dot{\underline{r}}_1|^2 + m_2 |\dot{\underline{r}}_2|^2 + m_3 |\dot{\underline{r}}_3|^2 \right) + \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} + \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} + \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad (5)$$

2) So:

$$\mathcal{L} = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) + \frac{m_1 m_2 G}{R_1} + \frac{m_1 m_3 G}{R_2} + \frac{m_2 m_3 G}{R_3} \quad -(6)$$

From eqs. (1) and (2):

$$m_1(\underline{R_1} + \underline{r_2}) + m_2\underline{r_2} + m_3\underline{r_3} = \underline{0} \quad -(7)$$

i.e.

$$\underline{r_2} = -\left(\frac{m_1\underline{R_1} + m_3\underline{r_3}}{m_1 + m_2}\right) \quad -(8)$$

From eqs. (1) and (2):

$$m_1\underline{r_1} + m_2(\underline{r_1} - \underline{R_1}) + m_3\underline{r_3} = \underline{0} \quad -(9)$$

$$\underline{r_1} = \frac{m_2\underline{R_1} - m_3\underline{r_3}}{m_1 + m_2} \quad -(10)$$

So:

$$r_1^2 = \frac{m_2^2 R_1^2 + m_3^2 r_3^2 - 2 m_2 m_3 \underline{R_1}\cdot\underline{r_3}}{(m_1 + m_2)^2} \quad -(11)$$

$$r_2^2 = \frac{m_1^2 R_1^2 + m_3^2 r_3^2 + 2 m_1 m_3 \underline{R_1}\cdot\underline{r_3}}{(m_1 + m_2)^2} \quad -(12)$$

Therefore:

$$m_1 r_1^2 + m_2 r_2^2 = \frac{(m_1 m_2^2 + m_2 m_1^2) R_1^2}{(m_1 + m_2)^2} + \frac{m_3^2}{m_1 + m_2} r_3^2 \quad -(13)$$

3)

$$= \mu R_1^2 + \mu_1 r_3^2$$

where:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_1 = \frac{m_3^2}{m_1 + m_2} \quad -(14)$$

Therefore:

$$\mathcal{L} = \frac{1}{2}\mu |\dot{\underline{R}}_1|^2 + m_3\left(1 + \frac{m_3}{m_1+m_2}\right)|\dot{\underline{r}}_3|^2 + \frac{m_1 m_2 G}{R_1} + \frac{m_1 m_3 G}{R_2} + \frac{m_2 m_3 G}{R_3} \quad -(15)$$

Here:

$$|\dot{\underline{R}}_1|^2 = \dot{R}_1^2 + R_1^2 \dot{\theta}_1^2 \quad -(16)$$

Now consider the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial R_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}_1} \quad -(17)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \quad -(18)$$

To obtain:

$$R_1 = \frac{\alpha_1}{1 + \epsilon_1 \cos\theta_1} \quad -(19)$$

Similarly:

4)

$$R_2 = \frac{\alpha_2}{1+\epsilon_2 \cos\theta_2} \quad -(20)$$

$$R_3 = \frac{\alpha_3}{1+\epsilon_3 \cos\theta_3} \quad -(21)$$

where

$$\underline{R_2} = \underline{R_1} + \underline{R_3} \quad -(22)$$

This is the same result as in note 221(1).

The general solution of the N particle problem is:

$$R_i = \frac{\alpha_i}{1+\epsilon_i \cos\theta_i}, \quad -(23)$$

$$i = 1, \ldots, N$$

constrained by a generalization of eq. (22).

For precessing orbits:

$$R_i = \frac{\alpha_i}{1+\epsilon_i \cos(x_i\theta_i)} \quad -(24)$$

$$i = 1, \ldots, N$$

Application of the Stokes theorem gives:

$$\oint \underline{R_i} \cdot d\underline{R_i} = \int \underline{S_i} \cdot \underline{n} \, dA_i \quad -(25)$$

) where $\underline{S}_i$ is the orbital circulation:

$$\underline{S}_i = \underline{\nabla} \times \underline{R}_i \qquad -(26)$$

$$= -\frac{1}{R_i}\frac{\partial R_i}{\partial \theta_i}\underline{k}$$

Here:

$$\frac{\partial R_i}{\partial \theta_i} = \frac{\epsilon_i}{d_i} R_i^2 \sin\theta_i . \qquad -(27)$$

So:

$$\underline{S}_i = -\frac{\epsilon_i}{d_i} R_i \sin\theta_i \, \underline{k}$$

$$\boxed{S_i = -\frac{\epsilon_i \sin\theta_i}{1+\epsilon_i \cos\theta_i}} \qquad -(28)$$

For precessing orbits:

$$S_i = -\frac{\epsilon_i x_i \sin(\theta_i x_i)}{1+\epsilon_i \cos(x_i \theta_i)} \qquad -(29)$$

In general:

$$dA_i = \frac{1}{2} R_i^2 \, d\theta_i \qquad -(30)$$

So:

$$\oint \underline{R}_i \cdot d\underline{R}_i = \frac{1}{2}\int S_i \, \underline{k} \cdot \underline{n} \, R_i^2 \, d\theta_i \qquad -(31)$$

Assuming:

$$\underline{k} \cdot \underline{n} = -1 \qquad -(32)$$

6) then:

$$\oint \underline{R_i} \cdot d\underline{R_i} = -\frac{1}{2} \int S_i R_i^2 \, d\theta_i$$

$$= \epsilon_i \alpha_i^2 \int \frac{\sin\theta_i \, d\theta_i}{(1 + \epsilon_i \cos\theta_i)^3} \quad -(33)$$

$$\boxed{\oint \underline{R_i} \cdot d\underline{R_i} = \epsilon_i \alpha_i^2 \int \frac{\sin\theta \, d\theta}{(1 + \epsilon_i \cos\theta)^3}} \quad -(34)$$

Note carefully that:

$$\oint \underline{R_i} \cdot d\underline{R_i} \neq \frac{R_i^2}{2}. \quad -(35).$$

For precessing orbits:

$$\oint \underline{R_i} \cdot d\underline{R_i} = \epsilon_i \alpha_i^2 x_i \int \frac{\sin(x_i \theta) \, d\theta}{(1 + \epsilon_i \cos(x_i \theta))^3} \quad -(36)$$

In the three particle problem:

$$\boxed{\oint \underline{R}_2 \cdot d\underline{R}_2 = \oint (\underline{R_1} + \underline{R_3}) \cdot d\underline{R_2}} \quad -(37)$$

Therefore from eq. (25):

$$\int S_2 \, dA_2 = \int (S_1 + S_3) \, dA_2 \quad -(26)$$

i.e

$$\underline{\nabla} \times \underline{R_2} = \underline{\nabla} \times \underline{R_1} + \underline{\nabla} \times \underline{R_3} \quad -(27)$$

So:

$$\boxed{\frac{\epsilon_1 \sin\theta_1}{1 + \epsilon_1 \cos\theta_1} = \frac{\epsilon_2 \sin\theta_2}{1 + \epsilon_2 \cos\theta_2} + \frac{\epsilon_3 \sin\theta_3}{1 + \epsilon_3 \cos\theta_3} \quad -(28)}$$

For each orbit of elapsed lines are:

$$t_i = \left(1 - \epsilon_i^2\right)^{3/2} \left(\frac{\tau_i}{2\pi}\right) \int \frac{d\theta}{\left(1 + \epsilon_i \cos\theta\right)^2} \quad -(29)$$

and

$$\epsilon_i = \left(1 - \left(\frac{a_i}{b_i}\right)^2\right)^{1/2} \quad -(30)$$

Therefore ϵ_i and θ_i, $i = 1, 2, 3$ can be measured experimentally and should obey of relation (28).