220(2): Total Linear Velocity and Comparison of Analytical and Numerical Methods.

The total linear velocity is given by:

$$v^2 = \dot{r}^2 + \dot{\theta}^2 r^2 \quad -(1)$$

where

$$\dot{\theta} = \frac{L}{mr^2} \quad -(2)$$

Therefore:

$$\boxed{v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2}} \quad -(3)$$

As in note 220(1):

$$\frac{dr}{dt} = \frac{L\epsilon}{m\alpha}\sin\theta \quad -(4)$$

so

$$v^2 = \frac{L^2}{m^2}\left(\left(\frac{\epsilon}{\alpha}\right)^2 \sin^2\theta + \frac{1}{r^2}\right) -(5)$$

where

$$\sin^2\theta = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 -(6)$$

Eqs (5) and (6) simplify as follows:

$$v^2 = \left(\frac{L}{m}\right)^2\left[\left(\frac{\epsilon}{\alpha}\right)^2\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right) + \frac{1}{r^2}\right]$$

$$= \left(\frac{L}{m}\right)^2\left[\left(\frac{\epsilon}{\alpha}\right)^2 + \frac{1}{r^2} - \frac{1}{\alpha^2}\left(\frac{\alpha}{r} - 1\right)^2\right]$$

2)

$$= \left(\frac{L}{m}\right)^2 \left[\left(\frac{\epsilon}{\alpha}\right)^2 + \frac{2}{\alpha r} - \frac{1}{\alpha^2}\right], \quad -(7)$$

where $$\alpha = \frac{L^2}{mk}, \quad \epsilon = \left(1 + \frac{2EL^2}{mk^2}\right)^{1/2}. \quad -(8)$$

Therefore:

$$v^2 = \left(\frac{L}{m}\right)^2 \left[\frac{m^2k^2}{L^4}\left(1 + \frac{2EL^2}{mk^2}\right) + \frac{2mk}{L^2 r} - \frac{m^2k^2}{L^4}\right]$$

$$= \left(\frac{L}{m}\right)^2 \left[\frac{m^2k^2}{L^4} \cdot \frac{2EL^2}{mk^2} + \frac{2mk}{L^2 r}\right]$$

$$= \frac{2E}{m} + \frac{2k}{mr} \quad -(9)$$

i.e. $$E = \frac{1}{2}mv^2 - \frac{k}{r} = \frac{1}{2}mv^2 - \frac{mMG}{r} \quad -(10)$$

QED.

However, it is also known that:

$$v^2 = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(11)$$

where:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(12)$$

for the ellipse and

$$a = \frac{\alpha}{\epsilon^2 - 1} \quad -(13)$$

3) for the hyperbola.

From eqs. (9) and (11):

$$\boxed{E = -\frac{k}{a}} \quad -(14)$$

<u>Application to Three Particle Problem</u>

From eq. (11):

$$v_1^2 = \frac{k_1}{\mu_1}\left(\frac{2}{R_1} - \frac{1}{a_1}\right) \quad -(15)$$

$$= \frac{2m_1m_2}{m_1m_2}(m_1+m_2)\left(\frac{2}{R_1} - \frac{1}{a_1}\right)$$

$$= 2(m_1+m_2)\left(\frac{2}{R_1} - \frac{1}{a_1}\right). \quad -(16)$$

From eq. (14):

$$a_1 = -\frac{k_1}{E_1} = -\frac{2m_1m_2}{E_1}$$

$$= \text{constant} \quad -(17)$$

So:

$$\boxed{\begin{aligned} v_1^2 &= 2(m_1+m_2)\left(\frac{2}{R_1} + \frac{E_1}{2m_1m_2}\right) \\ v_2^2 &= 2(m_1+m_3)\left(\frac{2}{R_2} + \frac{E_2}{2m_1m_3}\right) \\ v_3^2 &= 2(m_2+m_3)\left(\frac{2}{R_3} + \frac{E_3}{2m_2m_3}\right) \end{aligned}}$$

$-(18)$

4) These are three simultaneous equation in V_1, V_2 and V_3, given R_1, R_2 and R_3.

<u>Method 1</u>

Define the analytical parameters and derive the initial conditions. Eqs. (18) contain the velocities and positions of the centres of mass.

<u>Method 2</u>

Define the initial positions and velocities. These are given by $v_1(0)$, $v_2(0)$, $v_3(0)$, $R_1(0)$, $R_2(0)$ and $R_3(0)$. These define E_1, E_2 and E_3. The total energy of the three particle system is:

$$E = E_1 + E_2 + E_3 \quad -(19)$$

By definition:

$$\underline{R_1} = \underline{r_1} - \underline{r_2} \quad -(20)$$

$$\underline{R_2} = \underline{r_1} - \underline{r_3} \quad -(21)$$

$$\underline{R_3} = \underline{r_2} - \underline{r_3} \quad -(22)$$

where $\underline{r_1}$, $\underline{r_2}$ and $\underline{r_3}$ are the particle positions.

The analytical solution is uniquely defined by eqs. (18), which defines the initial conditions for the numerical solution.