220(B): Interrelation of Angles in the Three Particle Problem

Consider the mass of the sun, m_1, interacting with the Earth (m_2) and mass (m_3) as in Fig (1):

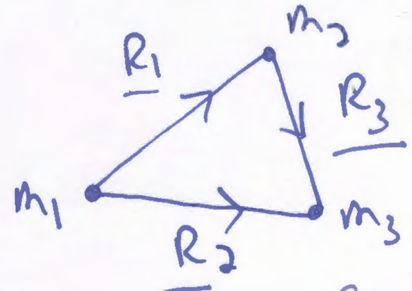

Fig (1)

The three vectors in a plane are related by:

$$\underline{R_2} = \underline{R_1} + \underline{R_3} \quad -(1)$$

In plane cylindrical coordinates the radial unit vector is

$$\underline{e}_r = \underline{i} \cos\theta + \underline{j} \sin\theta \quad -(2)$$

$$\underline{e}_\theta = -\underline{i} \sin\theta + \underline{j} \cos\theta. \quad -(3)$$

So:

$$\underline{R_2} = R_2 \underline{e}_r = R_{2x} \underline{i} + R_{2y} \underline{j}. \quad -(4)$$

In eq.(1)

$$R_2 \underline{e}_r = (R_1 + R_3) \underline{e}_r \quad -(5)$$

and

$$R_2 = R_1 + R_3. \quad -(6)$$

The orbit of the earth is described by:

$$R_2 = \frac{\alpha_2}{1 + \epsilon_2 \cos\theta_2} \quad -(7)$$

) and the orbit of Mars by:

$$R_3 = \frac{\alpha_3}{1+\epsilon_3 \cos\theta_3} \quad -(8)$$

The distance between earth and Mars is therefore:

$$R_3 = \frac{\alpha_2}{1+\epsilon_2 \cos\theta_2} - \frac{\alpha_1}{1+\epsilon_1 \cos\theta_1} \quad -(9)$$

So:

$$\cos\theta_2 = \frac{1}{\epsilon_2}\left[\frac{\alpha_2\left(1+\epsilon_1 \cos\theta_1\right)}{\alpha_1 + R_3\left(1+\epsilon_1 \cos\theta_1\right)} - 1\right] \quad -(10)$$

where:

$$\cos\theta_1 = \frac{1}{\epsilon_1}\left(\frac{\alpha_1}{R_1} - 1\right) \quad -(11)$$

therefore:

$$\boxed{\cos\theta_2 = \frac{1}{\epsilon_2}\left[\frac{\alpha_2}{R_1+R_3} - 1\right]} \quad -(12)$$

The orbit of the Earth around Mars is given by:

$$R_3 = R_2 - R_1 = \frac{\alpha_3}{1+\epsilon_3 \cos\theta_3} \quad -(13)$$

so the distance between Earth and Mars is affected by the gravitational attraction between

3) m_2 and m_3. This is a perturbation of the attraction result of the two particle problem, in which the attraction of the earth to the sun is considered, and the attraction of the planet Mars to the sun is considered as two independent two particle problems.

In the two particle problem:

$$R_3 = R_2 - R_1 \quad -(14)$$

is still true, but the result is:

$$R_3 = \frac{\alpha_2}{1+\epsilon_2 \cos\theta_2} - \frac{\alpha_1}{1+\epsilon_1 \cos\theta_1} \quad -(15)$$

In the three particle problem:

$$R_3 = \frac{\alpha_3}{1+\epsilon_3 \cos\theta_3} = \frac{\alpha_2}{1+\epsilon_2 \cos\theta_2} - \frac{\alpha_1}{1+\epsilon_1 \cos\theta_1} \quad -(16)$$

Therefore the relation between α_1 and α_2 and ϵ_1 and ϵ_2 is different in the 3 particle problem, defining eq. (15)

In the two particle problem the elapsed time of the orbits are

$$t_2 = \left(1-\epsilon_2^2\right)^{3/2} \frac{\tau_2}{2\pi} \int_0^{\theta_2} \frac{d\theta_2}{\left(1+\epsilon_2\cos\theta_2\right)^2},$$

$$t_1 = \left(1-\epsilon_1^2\right)^{3/2} \frac{\tau_1}{2\pi} \int_0^{\theta_1} \frac{d\theta_1}{\left(1+\epsilon_1\cos\theta_1\right)^2} \quad -(17)$$

4) The difference is:

$$\Delta t = t_2 - t_1 \qquad -(18)$$

Approximately:

$$\theta_1(t) = \frac{2\pi t}{\tau_1} + 2\epsilon_1 \sin\frac{2\pi t}{\tau_1} + \cdots \qquad -(19)$$

$$\theta_2(t) = \frac{2\pi t}{\tau_2} + 2\epsilon_2 \sin\frac{2\pi t}{\tau_2} + \cdots \qquad -(20)$$

For the situation considered in eq. (1) and Fig (1) it is convenient to consider the angle swept out in time t by the earth and Mars for example, or from the viewpoint of the earth by the sun m_1 and Mars m_3. In the three particle problem from the viewpoint of the sun, the earth sweeps out an angle θ_2 in time t and Mars sweeps out an angle θ_3 in time t. However, the earth will also sweep out an angle with respect to Mars. This does not occur in the two particle problem.

From eqs. (16), (19) and (20):

$$\frac{\alpha_3}{1+\epsilon_3\cos\theta_3} \sim \frac{\alpha_2}{1+\epsilon_2\cos\left(\frac{2\pi t}{\tau_1}\right)} - \frac{\alpha_1}{1+\epsilon_1\cos\left(\frac{2\pi t}{\tau_2}\right)} \qquad -(21)$$

giving θ_3 as a function of t.