220(9): Elapsed Time in ~~the~~ three Particle Problem

Consider the result of the three particle orbital problem:

$$\frac{\alpha_2}{1+\epsilon_2\cos\theta_2} = \frac{\alpha_1}{1+\epsilon_1\cos\theta_1} + \frac{\alpha_3}{1+\epsilon_3\cos\theta_3}. \quad -(1)$$

Note that:

$$\int \frac{\alpha_2}{1+\epsilon_2\cos\theta_2} d\theta_2 = \int R_2 \, d\theta_2 \quad -(2)$$

$$= \int \frac{\alpha_2}{1+\epsilon_2\cos\theta} d\theta = \int R_2 \, d\theta.$$

Therefore:

$$\int R_2 \, d\theta_2 = \int R_2 \, d\theta \quad -(3)$$

$$\int R_1 \, d\theta_1 = \int R_1 \, d\theta \quad -(4)$$

$$\int R_3 \, d\theta_3 = \int R_3 \, d\theta \quad -(5)$$

So:

$$R_2 \, d\theta_2 = R_2 \, d\theta \quad -(6)$$

$$R_3 \, d\theta_3 = R_3 \, d\theta$$

$$R_1 \, d\theta_1 = R_1 \, d\theta$$

i.e.

$$d\theta = d\theta_1 = d\theta_2 = d\theta_3. \quad -(7)$$

Therefore:

$$\int \frac{\alpha_2 \, d\theta_2}{1+\epsilon_2\cos\theta_2} = \int \frac{\alpha_2 \, d\theta}{1+\epsilon_2\cos\theta} \quad -(8)$$

$$= \int \left(\frac{\alpha_1}{1+\epsilon_1\cos\theta} + \frac{\alpha_3}{1+\epsilon_3\cos\theta} \right) d\theta.$$

The elapsed time is given by the expression:

$$\frac{\pi ab}{\tau} t = \int dA = \frac{1}{2}\int_0^\theta r^2 \, d\theta \quad -(9)$$

So:

$$t = \frac{\tau}{2\pi ab}\int_0^\theta \left(\frac{\alpha_2}{1+\epsilon_2\cos\theta}\right)^2 d\theta \quad -(10)$$

$$= \left(\frac{\tau_2 \alpha_2^2}{2\pi a_2 b_2}\right)\left[\frac{\epsilon_2 \sin\theta}{(\epsilon_2^2-1)(1+\epsilon_2\cos\theta)} + \frac{1}{(1-\epsilon_2^2)}\int_0^\theta \frac{d\theta}{1+\epsilon_2\cos\theta}\right]$$

$$t = \frac{\tau}{2\pi ab}\left[\frac{\alpha_2}{(1-\epsilon_2^2)}\int_0^\theta \frac{\alpha_2 \, d\theta}{1+\epsilon_2\cos\theta} - \frac{\alpha_2^2\epsilon_2\sin\theta}{(1-\epsilon_2^2)(1+\epsilon_2\cos\theta)}\right]$$

where:

$$\int_0^\theta \frac{\alpha_2 \, d\theta}{1+\epsilon_2\cos\theta} = \int_0^\theta \left(\frac{\alpha_1}{1+\epsilon_1\cos\theta} + \frac{\alpha_3}{1+\epsilon_3\cos\theta}\right) d\theta \quad -(11)$$

Finally:

$$\int_0^\theta \frac{\alpha_1 \, d\theta}{1+\epsilon_1\cos\theta} = \frac{2\alpha_1}{(1-\epsilon_1^2)^{1/2}}\tan^{-1}\left(\frac{(1-\epsilon_1)\tan(\theta/2)}{(1-\epsilon_1^2)^{1/2}}\right)$$

$$\int_0^\theta \frac{\alpha_3 \, d\theta}{1+\epsilon_3\cos\theta} = \frac{2\alpha_3}{(1-\epsilon_3^2)^{1/2}}\tan^{-1}\left(\frac{(1-\epsilon_3)\tan(\theta/2)}{(1-\epsilon_3^2)^{1/2}}\right).$$

$$-(12)$$

3) Therefore the elapsed time t of the orbit of, for example, the earth of half right latitude α_2 and eccentricity of ϵ_2 is affected by the orbit of Mars about the sun and the orbit of Mars about the earth. Here:

$$a_2 b_2 = \frac{\alpha_2^2}{(1-\epsilon_2^2)^{3/2}} \quad -(13)$$

$$-(14)$$

So in eq. (10):

$$t = \frac{\tau_2}{2\pi}\left(1-\epsilon_2^2\right)^{-3/2}\int_0^\theta \frac{d\theta}{(1+\epsilon_2\cos\theta)^2}$$

which reduces the problem to the experimentally observable τ_2 and ϵ_2.

So:

$$t = \frac{\tau_2}{2\pi}\left(1-\epsilon_2^2\right)^{-3/2}\left[\frac{\epsilon_2\sin\theta}{(\epsilon_2^2-1)(1+\epsilon_2\cos\theta)} + \frac{1}{1-\epsilon_2^2}\int_0^\theta \frac{d\theta}{1+\epsilon_2\cos\theta}\right] \quad -(15)$$

where:

$$\int_0^\theta \frac{d\theta}{1+\epsilon_2\cos\theta} = \frac{1}{\alpha_2}\int_0^\theta\left(\frac{\alpha_1}{1+\epsilon_1\cos\theta} + \frac{\alpha_3}{1+\epsilon_3\cos\theta}\right)d\theta \quad (16)$$

Finally:

4)
$$d_i = a_i\left(1-\epsilon_i^2\right), \quad — (17)$$

$i = 1, 2, 3$

where a_i is the semi major axis of each orbit. The quantities a_i, ϵ_i and τ_i are observed experimentally.