220(7): Elapsed Time in a Precessing Two Particle Orbit.

In the two particle orbit:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \quad -(1)$$

where
$$U(r) = -\frac{mMGx^2}{r} - \frac{L^2}{2mr^2}\left(1-x^2\right), \quad -(2)$$

and
$$L = \mu r^2 \frac{d\theta}{dt}. \quad -(3)$$

Therefore the orbit is:
$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)}. \quad -(4)$$

The area of the orbit is:
$$A = \frac{1}{2}\int r^2\, d\theta. \quad -(5)$$

Therefore:
$$A = \frac{1}{2}\int\left(\frac{\alpha}{1+\epsilon\cos(x\theta)}\right)^2 d\theta. \quad -(6)$$

Eq. (5) is the integral form of the general result for any curve in two dimensions:
$$dA = \frac{1}{2}r^2\, d\theta. \quad -(7)$$

From eq. (7):
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{L}{2\mu} \quad -(8)$$
$$= \text{constant}$$

2) This is Kepler's second law.

From this law:

$$\int dt = \frac{2\mu}{L} \int dA \qquad -(9)$$

Denote:

$$\tau = \int dt, \quad A = \int dA, \qquad -(10)$$

then

$$\boxed{\tau = \frac{2\mu A}{L}} \qquad -(11)$$

The elapsed time τ can be measured to great accuracy as a function of θ in modern astronomy.

To integrate eq. (6) analytically use:

$$\beta = x\theta \qquad -(12)$$

then

$$d\theta = \frac{1}{x} d\beta \qquad -(13)$$

so:

$$A = \frac{1}{2x} \int \left(\frac{\alpha}{1+\epsilon \cos\beta} \right)^2 d\beta \qquad -(14)$$

$$= \frac{\alpha^2}{2x} \int \frac{1}{(1+\epsilon\cos\beta)^2} d\beta \qquad -(15)$$

The integral is:

$$\int \frac{1}{(1+\epsilon\cos\beta)^2}\, d\beta \qquad -(16)$$

$$= \frac{2}{(1-\epsilon^2)^{3/2}} \tan^{-1}\left((1-\epsilon^2)^{1/2}\tan\frac{\beta}{2}\right) - \frac{\epsilon\sin\beta}{(1-\epsilon^2)(1+\epsilon\cos\beta)}$$

Therefore the elapsed time of the precessing orbit is:

$$\tau = \frac{2\mu}{L} A, \qquad -(17)$$

$$A = \frac{\alpha^2}{2x}\left[\frac{2}{(1-\epsilon^2)^{3/2}}\tan^{-1}\left((1-\epsilon^2)^{1/2}\tan\left(\frac{x\theta}{2}\right)\right) - \frac{\epsilon\sin(x\theta)}{(1-\epsilon^2)(1+\epsilon\cos(x\theta))}\right]$$

$$-(18)$$

Graphics

Plot τ as a function of θ for given x.

The static ellipse is given by:

$$x = 1 \qquad -(19)$$

The area of the static ellipse is:

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(1+\cos\theta)^2} \qquad -(20)$$

$$= \frac{\pi}{(1-\epsilon^2)^{3/2}} = \pi ab$$

4)

where a and b are the semimajor and semiminor axes. Therefore for the static ellipse:

$$\tau = \frac{2\pi\mu}{L} ab. \quad -(21)$$

The static ellipse corresponds to the Newtonian theory, where:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|} \quad -(22)$$

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2\mu|E|)^{1/2}} \quad -(23)$$

$$= (\alpha a)^{1/2},$$

$$\alpha = \frac{L^2}{\mu k} \quad -(24)$$

So:

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3 \quad -(25)$$

which is Kepler's third law.

However for the precessing ellipse (4) Kepler's third law no longer holds. It is replaced by eq. (17).

Thus:

5)

$$\tau = \frac{\mu \alpha^2}{x L} f(\theta) \quad -(26)$$

where:

$$f(\theta) = \frac{2}{(1-\epsilon^2)^{3/2}} \tan^{-1}\left((1-\epsilon^2)^{1/2} \tan\left(\frac{x\theta}{2}\right)\right) - \frac{\epsilon \sin(\theta x)}{(1-\epsilon^2)(1+\epsilon\cos(x\theta))} \quad -(27)$$

From eq. (24):

$$\boxed{\tau = \frac{\left(a(1-\epsilon^2)\right)^{3/2}}{x k} f(\theta)} \quad -(28)$$

where

$$k = m M G \quad -(29)$$

Eq. (28) generalizes Kepler's Third Law for the precessing ellipse. The third law is recovered when:

$$x = 1. \quad -(30)$$

It is seen that the time τ taken for an angle $\theta = 2\pi$ is different for a precessing ellipse.