Note 220(1): N Particle Problem, Expressions for Position and Velocity

Express the three orbits of the centres of mass as:

$$R_i = \frac{\alpha_i}{1+\epsilon_i \cos\theta_i}, \quad -(1)$$

$$i = 1, 2, 3,$$

where:

$$\alpha_i = \frac{L_i^2}{\mu_i k_i}, \quad \epsilon_i = \left(1 + \frac{2E_i L_i^2}{\mu_i k_i^2}\right)^{1/2}, \quad -(2)$$

$$L_i = \mu_i R_i^2 \frac{d\theta_i}{dt}, \quad -(3)$$

$$k_1 = 2m_1m_2, \quad k_2 = 2m_1m_3, \quad k_3 = 2m_2m_3. \quad -(4)$$

From eqs. (1) and (3):

$$\frac{d\theta_1}{dt} = \frac{L_1}{\mu_1 R_1^2} = \frac{L_1}{\mu_1 \alpha_1^2}(1+\epsilon_1\cos\theta_1)^2 \quad -(5)$$

$$\frac{d\theta_2}{dt} = \frac{L_2}{\mu_2 R_2^2} = \frac{L_2}{\mu_2 \alpha_2^2}(1+\epsilon_2\cos\theta_2)^2 \quad -(6)$$

$$\frac{d\theta_3}{dt} = \frac{L_3}{\mu_3 R_3^2} = \frac{L_3}{\mu_3 \alpha_3^2}(1+\epsilon_3\cos\theta_3)^2 \quad -(7)$$

Now use:

$$\frac{d\theta_i}{dt} = \frac{d\theta_i}{dR_i}\frac{dR_i}{dt} \quad -(8)$$

so:

$$\frac{dR_i}{dt} = \frac{dR_i}{d\theta_i}\frac{d\theta_i}{dt} \quad -(9)$$

Here:

$$\frac{dR_i}{d\theta_i} = \frac{\epsilon_i \alpha_i \sin\theta_i}{(1+\epsilon_i \cos\theta_i)^2} \quad -(10)$$

$$\frac{d\theta_i}{dt} = \frac{L_i}{\mu_i \alpha_i^2}\left(1+\epsilon_i \cos\theta_i\right)^2 \quad -(11)$$

So

$$\frac{dR_i}{dt} = \frac{L_i \epsilon_i \alpha_i}{\mu_i \alpha_i^2} \sin\theta_i \quad -(12)$$

$$\boxed{\begin{aligned} \frac{dR_i}{dt} &= \left(\frac{L_i \epsilon_i}{\mu_i \alpha_i}\right) \sin\theta_i \quad -(13) \\ R_i &= \frac{\alpha_i}{1+\epsilon_i \cos\theta_i} \quad -(14) \end{aligned}}$$

In these equations:

$$\sin\theta_i = \left[1 - \frac{1}{\epsilon_i^2}\left(\frac{\alpha_i}{R_i} - 1\right)^2\right]^{1/2} \quad -(15)$$

So there are three equations:

$$\boxed{\frac{dR_i}{dt} = \left(\frac{L_i \epsilon_i}{\mu_i \alpha_i}\right)\left[1 - \frac{1}{\epsilon_i^2}\left(\frac{\alpha_i}{R_i} - 1\right)^2\right]^{1/2}} \quad -(16)$$