220(8): Summary of Theory for the Two Particle and N Particle Problems.

In the two particle problem:

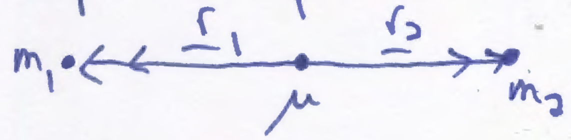


Fig (1)

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m_1 |\dot{\underline{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\underline{r}}_2|^2 - U(R) \quad -(1)$$

where

$$\underline{R} = \underline{r}_1 - \underline{r}_2 . \quad -(2)$$

The centre of mass of m_1 and m_2 is defined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0} . \quad -(3)$$

Therefore:

$$\underline{r}_1 = \frac{m_2}{m_1 + m_2} \underline{R} , \quad \underline{r}_2 = - \frac{m_1}{m_1 + m_2} \underline{R} . \quad -(4)$$

Therefore:

$$\mathcal{L} = \frac{1}{2} \mu |\dot{\underline{R}}|^2 - U(R) \quad -(5)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} . \quad -(6)$$

In plane cylindrical coordinates:

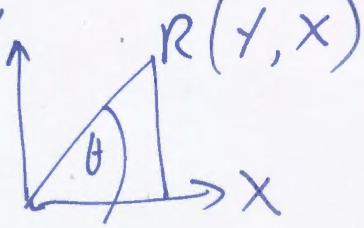
$$\dot{\underline{R}} = \dot{R} \underline{e}_r + R \dot{\underline{e}}_r \quad -(7)$$

$$= \dot{R} \underline{e}_r + R \dot{\theta} \underline{e}_\theta$$

Here:

$$\underline{R} = X \underline{i} + Y \underline{j} = R \cos\theta \, \underline{i} + R \sin\theta \, \underline{j} \quad -(8)$$

) Here:



$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta$$
$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta.$$
$$\underline{i} = \underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta$$
$$\underline{j} = \underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta$$

So:

$$\begin{aligned}\underline{R} &= X\underline{i} + Y\underline{j}\\ &= R\cos\theta\left(\underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta\right) + R\sin\theta\left(\underline{e}_r\sin\theta + \underline{e}_\theta\sin\theta\right)\\ &= R\underline{e}_r\left(\cos^2\theta + \sin^2\theta\right) \quad -(9)\\ &= R\underline{e}_r.\end{aligned}$$

The time derivative of $\underline{e}_r$ is:

$$\dot{\underline{e}}_r = \frac{d\underline{e}_r}{dt} = \frac{d}{dt}\left(\underline{i}\cos\theta(t) + \underline{j}\sin\theta(t)\right) - (10)$$

so:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \quad -(11)$$
$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r. \quad -(12)$$

Therefore:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{R}^2 + R^2\dot{\theta}^2\right) - U(R). \quad -(13)$$

The Euler Lagrange equation is:

$$\frac{\partial\mathcal{L}}{\partial q_j} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_j}, \quad -(14)$$
$$j = 1, 2, \ldots, s$$

3) and the Lagrangian is:

$$\mathcal{L} = T(q_j, \dot{q}_j, t) - U(q_j, t) \quad -(15)$$

Here q_j are the generalized coordinates, a set of quantities that specify completely the state of the system, with number of degrees of freedom S.

For the Lagrangian (13):

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(16)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}. \quad -(17)$$

These reduce to:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L^2} F(r), \quad -(18)$$

where

$$F(r) = -\frac{\partial U(r)}{\partial r} \quad -(19)$$

and

$$L = \mu r^2 \frac{d\theta}{dt} \quad -(20)$$

For a Hooke Newton potential:

$$U(r) = -m_1 m_2 G / r \quad -(21)$$

the orbit is the static ellipse:

4)

$$r = \frac{\alpha}{1+\epsilon\cos\theta}, \quad -(22)$$

and more generally the static conical section. Here:

$$L^2 = \alpha \mu k \quad -(23)$$

where

$$k = m_1 m_2 G \quad -(24)$$

and

$$\alpha = a(1-\epsilon^2) \quad -(25)$$

where a is the semi major axis.

The time taken for a mass to sweep out an angle θ is:

$$\tau = \frac{\alpha^{3/2}}{m_1 m_2 G} f(\theta) \quad -(26)$$

where:

$$-(27)$$

$$f(\theta) = \int \frac{1}{(1+\cos\theta)^2} d\theta$$

$$= \frac{2}{(1-\epsilon^2)^{3/2}} \tan^{-1}\left((1-\epsilon^2)^{1/2}\tan\frac{\theta}{2}\right) - \frac{\epsilon\sin\theta}{(1-\epsilon^2)(1+\epsilon\cos\theta)}$$

The area of the ellipse is:

$$A = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(1+\cos\theta)^2} = \pi ab. \quad -(28)$$

In time τ, the angle has swept out 2π radians, covering the entire area πab of the static ellipse. In a time t, an area $(\pi ab/\tau)t$ is swept out. So:

$$\frac{\pi ab}{\tau} t = \int dA = \frac{1}{2}\int_0^\theta r^2 \, d\theta. \quad -(29)$$

Here

$$ab = \frac{\alpha^2}{(1-\epsilon^2)^{3/2}} \quad -(30)$$

so:

$$t = \frac{\tau}{2\pi}\left[2\tan^{-1}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}\tan\frac{\theta}{2}\right) - \frac{\epsilon(1-\epsilon^2)^{1/2}\sin\theta}{1+\epsilon\cos\theta}\right] \quad -(31)$$

This is the time taken for θ to change from zero to θ. It is expressed in terms of the fundamental observables of the orbit: τ and ϵ, and can be measured with great accuracy in modern astronomy. Using computer algebra eq. (31) can be inverted to give:

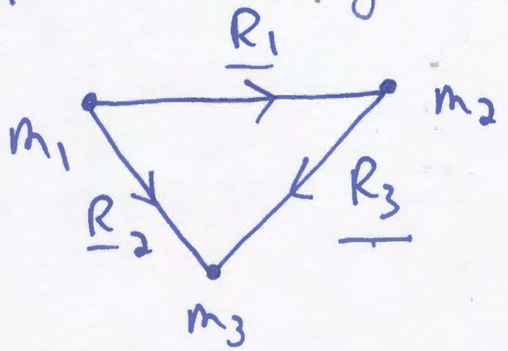
$$\theta = f(t) \quad -(32)$$

to machine precision. Approximately:

$$\theta(t) = 2\pi\frac{t}{\tau} + 2\epsilon\sin\frac{2\pi t}{\tau} + \frac{5}{4}\epsilon^2\sin\frac{4\pi t}{\tau} + \frac{1}{12}\epsilon^3\left(13\sin\frac{6\pi t}{\tau} - 3\sin\frac{2\pi t}{\tau}\right) + \ldots \quad -(33)$$

6) <u>Three Particle Problem</u>

This is sketched in Fig (2):



so:

$$\underline{R}_2 = \underline{R}_1 + \underline{R}_3 . \quad - (34)$$

Here:

$$\underline{R}_i = R_i \, \underline{e}_r , \quad - (35)$$

$$i = 1, 2, 3$$

so

$$R_2 = R_1 + R_3 . \quad - (36)$$

Here:

$$R_1 = | \underline{r}_1 - \underline{r}_2 | \quad - (37)$$

$$R_2 = | \underline{r}_1 - \underline{r}_3 | \quad - (38)$$

$$R_3 = | \underline{r}_2 - \underline{r}_3 | \quad - (39)$$

and

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0} \quad - (40)$$

$$m_1 \underline{r}_1 + m_3 \underline{r}_3 = \underline{0} \quad - (41)$$

$$m_2 \underline{r}_2 + m_3 \underline{r}_3 = \underline{0} \quad - (42)$$

define three centres of mass.

The three particle problem means that m_1 interacts <u>simultaneously</u> with the fields of m_2 and m_3 and so on.

7) The three particle lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} - \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} - \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad -(43)$$

and is factorized as follows:

$$\mathcal{L}_1 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2\right) - \frac{2m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} \quad -(44)$$

$$\mathcal{L}_2 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{2m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} \quad -(45)$$

$$\mathcal{L}_3 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{2m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad -(46)$$

Each one of these has the same format as eq. (1). Therefore the three inter-
of lagrangians (44) to (46) generate the method of rotated elliptical orbits. Following the factorized eq. (1) and following equations, the lagrangian can be written as:

$$\mathcal{L}_1 = \frac{1}{2}\mu_1|\dot{\underline{R}}_1|^2 - U(R_1) \quad -(47)$$

$$\mathcal{L}_2 = \frac{1}{2}\mu_2|\dot{\underline{R}}_2|^2 - U(R_2) \quad -(48)$$

$$\mathcal{L}_3 = \frac{1}{2}\mu_3|\dot{\underline{R}}_3|^2 - U(R_3) \quad -(49)$$

in the format of eq. (5). Here:

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 = \frac{m_1 m_3}{m_1 + m_3}, \quad \mu_3 = \frac{m_2 m_3}{m_2 + m_3}, \quad -(50)$$

$$U_1 = -\frac{2m_1m_2G}{R_1}, \quad U_2 = -\frac{2m_1m_3G}{R_2}, \quad U_3 = -\frac{2m_2m_3G}{R_3}. \quad -(51)$$

We define:

$$k_1 = 2m_1m_2G, \quad k_2 = 2m_1m_3G, \quad k_3 = 2m_2m_3G. \quad -(53)$$

In plane cylindrical coordinates, eqs. (47) to (49) are:

$$\mathcal{L}_1 = \frac{1}{2}\mu_1\left(\dot{R}_1^2 + R_1^2\dot{\theta}_1^2\right) - U(R_1) \quad -(54)$$

$$\mathcal{L}_2 = \frac{1}{2}\mu_2\left(\dot{R}_2^2 + R_2^2\dot{\theta}_2^2\right) - U(R_2) \quad -(55)$$

$$\mathcal{L}_3 = \frac{1}{2}\mu_3\left(\dot{R}_3^2 + R_3^2\dot{\theta}_3^2\right) - U(R_3) \quad -(56)$$

Each lagrangian is governed by Euler Lagrange equations of type (14). So:

$$\frac{\partial \mathcal{L}_i}{\partial R_i} = \frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{R}_i}, \quad -(57)$$

$$\frac{\partial \mathcal{L}_i}{\partial \theta_i} = \frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{\theta}_i}, \quad -(58)$$

$$i = 1, 2, 3.$$

So there are three equations of type (18):

$$\frac{d^2}{d\theta_i^2}\left(\frac{1}{R_i}\right) + \frac{1}{R_i} = -\frac{\mu_i R_i^2 F_i(R_i)}{L_i^2} \quad -(59)$$

$$i = 1, 2, 3$$

), Here:

$$F_1 = -\frac{2 m_1 m_2 G}{R_1^2}, \quad F_2 = -\frac{2 m_1 m_3 G}{R_2^2}, \quad F_3 = -\frac{2 m_2 m_3 G}{R_3^2}$$

$$L_1^2 = \alpha_1 \mu_1 k_1, \quad L_2^2 = \alpha_2 \mu_2 k_2, \quad -(60)$$

$$L_3^2 = \alpha_3 \mu_3 k_3. \quad -(61)$$

So there are three inter-related orbits:

$$R_i = \frac{\alpha_i}{1+\epsilon_i \cos\theta_i}, \quad -(62)$$

$$i = 1, 2, 3.$$

From eqs. (36) and (62):

$$\frac{\alpha_2}{1+\epsilon_2\cos\theta_2} = \frac{\alpha_1}{1+\epsilon_1\cos\theta_1} + \frac{\alpha_3}{1+\epsilon_3\cos\theta_3} \quad -(63)$$

If for example m_1 is the sun, m_2 is the earth and m_3 is Mars, then the orbit of Mars about the earth is given by:

$$\frac{\alpha_3}{1+\epsilon_3\cos\theta_3} = \frac{\alpha_2}{1+\epsilon_2\cos\theta_2} - \frac{\alpha_1}{1+\epsilon_1\cos\theta_1}. \quad -(64)$$

The orbit of the earth about the sun is given by:

$$\frac{\alpha_1}{1+\epsilon_1\cos\theta_1} = \frac{\alpha_2}{1+\epsilon_2\cos\theta_2} - \frac{\alpha_3}{1+\epsilon_3\cos\theta_3} \quad -(65)$$

The orbit of Mars about the sun is given by eq. (63).

In order to evaluate orbital elapsed time we use the result:

$$\int\frac{d\theta}{(1+\epsilon\cos\theta)^2} = \frac{\epsilon\sin\theta}{(\epsilon^2-1)(1+\epsilon\cos\theta)} - \left(\frac{\epsilon}{1-\epsilon^2}\right)\int\frac{d\theta}{1+\epsilon\cos\theta} \quad -(66)$$

From eq. (65):

$$\int\frac{\alpha_1}{1+\epsilon_1\cos\theta_1}\,d\theta_1 = \int\frac{\alpha_1}{1+\epsilon_1\cos\theta}\,d\theta$$

$$= \int\frac{\alpha_2\,d\theta_1}{1+\epsilon_2\cos\theta_2} - \int\frac{\alpha_3\,d\theta_1}{1+\epsilon_3\cos\theta_3} \quad -(67)$$

so:

$$\int\frac{\alpha_1\,d\theta}{(1+\epsilon_1\cos\theta)^2} = \frac{\alpha_1\epsilon_1\sin\theta}{(\epsilon_1^2-1)(1+\epsilon_1\cos\theta)} - \frac{\epsilon_1}{1-\epsilon_1^2}\left(\int\frac{\alpha_2\,d\theta_1}{1+\epsilon_2\cos\theta_2} - \int\frac{\alpha_3\,d\theta_1}{1+\epsilon_3\cos\theta_3}\right) \quad -(68)$$

It is seen that the orbital elapsed time of the earth about the sun is affected by the other two orbits. The next note will evaluate this.