220(5): Evaluation of Time Elapsed for a Precessing Ellipse.

The time taken for the ellipse (i.e. r) to sweep through an angle θ is:

$$t = (1-\epsilon^2)^{3/2} \frac{\tau}{2\pi} \int_0^\theta \frac{d\theta}{(1+\epsilon\cos(\theta))^2}. \quad -(1)$$

Here τ is the time needed for r to sweep out the area of the ellipse, πab. The area swept out in time t is $(\pi ab/\tau)t$:

$$\frac{\pi ab}{\tau} t = \int dA \quad -(2)$$

where

$$dA = \frac{1}{2} r^2 d\theta. \quad -(3)$$

At $t=0$, $\theta = 0$, so:

$$\frac{\pi ab}{\tau} t = \frac{1}{2}\int_0^\theta r^2 d\theta \quad -(4)$$

where

$$r^2 = \frac{d^2}{(1+\epsilon\cos(\theta))^2}, \quad -(5)$$

$$ab = \frac{d^2}{(1-\epsilon^2)^{3/2}}, \quad -(6)$$

so we obtain eq. (1), QED.

In order to evaluate eq. (1)

2) use standard integrals to obtain:

$$t = \frac{\tau}{2\pi}\left[2\tan^{-1}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}\tan\frac{\theta}{2}\right) - \epsilon\frac{\left(1-\epsilon^2\right)^{1/2}\sin\theta}{1+\epsilon\cos\theta}\right] \quad -(7)$$

If the ellipse is precessing then:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(8)$$

so

$$\boxed{\theta \rightarrow x\theta} . \quad -(9)$$

the angle of precession is:

$$\Delta\theta = x\theta - \theta \quad -(10)$$

Making the substitution (9) in eq. (1) then:

$$t_1 = \left(1-\epsilon^2\right)^{3/2}\frac{\tau}{2\pi}\int_0^{\beta}\frac{d\beta}{\left(1+\epsilon\cos\beta\right)^2} \quad -(11)$$

$$= \frac{\tau}{2\pi}\left[2\tan^{-1}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}\tan\left(\frac{x\theta}{2}\right)\right) - \epsilon\frac{\left(1-\epsilon^2\right)^{1/2}\sin(x\theta)}{1+\epsilon\cos(x\theta)}\right]$$

which is the time taken to sweep out an angle β.

It will be interesting to plot t against θ for eq. (7), and t_1 against $x\theta$ for eq. (11)