219(8): Representation of a Three Dimensional Function by a Generalized Conical Section.

The problem can be stated as:

$$f(r, \theta, Z) = \frac{1}{r} = \frac{1}{d}\left(1 + \epsilon \cos(x\theta)\right) \quad -(1)$$

where:

$$x = x(\theta, Z). \quad -(2)$$

From eq. (1):

$$\frac{df}{d\theta} = -\frac{\epsilon}{d}\left(x + \theta \frac{dx}{d\theta}\right) \sin(\theta x) \quad -(3)$$

$$\frac{df}{dZ} = -\frac{\epsilon}{d}\left(x \frac{d\theta}{dZ} + \theta \frac{dx}{dZ}\right) \sin(\theta x) \quad -(4)$$

In eq. (4):

$$\frac{d\theta}{dZ} = \frac{d\theta}{dx} \frac{dx}{dZ} \quad -(5)$$

From eq. (1):

$$1 + \epsilon \cos(x\theta) = \frac{d}{r}, \quad -(6)$$

so

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{d}{r} - 1\right), \quad -(7)$$

and

$$\sin(x\theta) = \left(1 - \cos^2(x\theta)\right)^{1/2}. \quad -(8)$$

The dependence of x on θ can be found from eq. (3) by:

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$$x + \theta \frac{dx}{d\theta} = -\frac{\alpha}{\epsilon \sin(\theta x)} \frac{df}{d\theta} \quad -(9)$$

This is a differential equation in x. Similarly the dependence of x on z can be found by:

$$x \frac{d\theta}{dz} + \theta \frac{dx}{dz} = -\frac{\alpha}{\epsilon \sin(\theta x)} \frac{df}{dz}, \quad -(10)$$

using eq. (5). In addition there are the equations:

$$\frac{dr}{d\theta} = -r^2 \frac{df}{d\theta} \quad -(11)$$

and

$$\frac{dr}{dz} = -r^2 \frac{df}{dz} \quad -(12)$$

therefore given the function $f(r, \theta, z)$, the functions $df/d\theta$ and df/dz can be found. These are used in eqs. (9) and (10), in which:

$$\frac{1}{\sin(x\theta)} = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2} \quad -(13)$$

Denote:

$$A = \frac{df}{d\theta}, \quad B = \frac{df}{dz} \quad -(14)$$

then:

$$x + \theta \frac{dx}{d\theta} = -\frac{\alpha A}{\epsilon}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2} \quad -(15)$$

This equation must be integrated numerically.

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From eqs. (5) and (12):

$$\frac{d\theta}{dZ} = -r^2 \frac{d\theta}{dr} \cdot \frac{df}{dZ} = -r^2 \frac{d\theta}{dr} B \quad -(16)$$

From eq. (11):

$$\frac{d\theta}{dr} = -\frac{A}{r^2} \quad -(17)$$

so

$$\frac{d\theta}{dZ} = -AB \quad -(18)$$

In eq. (10):

$$-ABx + \theta \frac{dx}{dZ} = -\frac{d}{\epsilon \sin(\theta x)} \frac{df}{dZ}$$

$$= -\frac{dB}{\epsilon}\left(1 - \frac{1}{\epsilon^2}\left(\frac{d}{r} - 1\right)^2\right)^{-1/2} \quad -(19)$$

This equation must be integrated numerically to find the dependence of x on Z.