219 (6): Three Dimensional 3 - Particle Lagrangian and Motion of Two planets in a Plane.

In general for three particles the 3 - D lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_3 \dot{r}_3^2\right) - \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} \quad -(1)$$

$$- \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} - \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad -(2)$$

where: $$\underline{\nabla}\left(\frac{1}{|\underline{r} - \underline{a}|}\right) = - \frac{\underline{r} - \underline{a}}{|\underline{r} - \underline{a}|^3} .$$

Here: $$\underline{r} - \underline{a} = (x - a_x)\underline{i} + (y - a_y)\underline{j} + (z - a_z)\underline{k} \quad -(3)$$

and $$|\underline{r} - \underline{a}| = \left((x - a_x)^2 + (y - a_y)^2 + (z - a_z)^2\right)^{1/2} . \quad -(4)$$

When two planets orbit the sun in a plane:

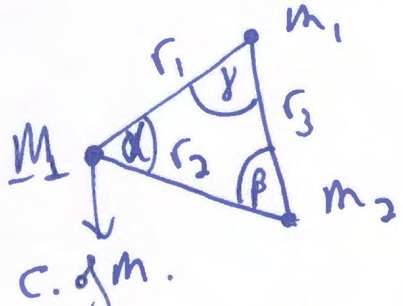

Fig (1)

the lagrangian can be written in terms of r_1, r_2 and r_3. These are related by:

$$r_3^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos \alpha \quad -(5)$$

$$r_1^2 = r_2^2 + r_3^2 - 2 r_2 r_3 \cos \beta$$

$$r_2^2 = r_1^2 + r_3^2 - 2 r_1 r_3 \cos \gamma$$

2) The lagrangian is, for Fig (1):

$$\mathcal{L} = \frac{1}{2}\left(\frac{m_1 M}{m_1+M}\dot{r}_1^2 + \frac{m_2 M}{m_2+M}\dot{r}_2^2 + \frac{m_1 m_2}{m_1+m_2}\dot{r}_3^2 - \frac{m_1 M G}{r_1} - \frac{m_2 M G}{r_2} - \frac{m_1 m_2 G}{r_3}\right) \quad -(6)$$

in the Newtonian limit. The three masses M, m_1 and m_2 are in the same plane. The four Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_1} \quad -(7)$$

$$\frac{\partial \mathcal{L}}{\partial r_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_2} \quad -(8)$$

$$\frac{\partial \mathcal{L}}{\partial r_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_3} \quad -(9)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(10)$$

Using the methods of previous notes eqs. (6) to (10) give:

$$r_1 = \frac{\alpha_1}{1+\epsilon_1 \cos\theta} \quad -(11)$$

$$r_2 = \frac{\alpha_2}{1+\epsilon_2 \cos\theta} \quad -(12)$$

$$r_3 = \frac{\alpha_3}{1+\epsilon_3 \cos\theta} \quad -(13)$$

) So the solution of the problem is given by eqs. (5), (11), (12) and (13). By definition:

$$\alpha_1 = \frac{L_1^2}{\mu_1 k_1} \; ; \; \epsilon_1 = \left(1 + \frac{2 E_1 L_1^2}{\mu_1 k_1^2}\right)^{1/2} \; ; \; k_1 = m_1 M G \; ; \quad -(14)$$

$$L_1 = \mu_1 r_1^2 \frac{d\theta}{dt}, \quad -(15)$$

$$\mu_1 = \frac{m_1 M}{m_1 + M} \sim m_1 \quad -(16)$$

$$E_1 = -\frac{m_1 M G}{2 a_1} \quad -(17)$$

$$\alpha_1 = a_1 (1 - \epsilon_1^2),$$

and so on.

If the reduced mass of m_1 and m_2 is:

$$\mu_3 = \frac{m_1 m_2}{m_1 + m_2} \quad -(18)$$

then the problem can be stated as the motion of μ_3 around M. Define the reduced mass:

$$\mu_4 = \frac{\mu_3 M}{\mu_3 + M} \quad -(19)$$

then the lagrangian is:

$$\mathcal{L} = \frac{1}{2} \mu_4 \dot{r}^2 - \frac{\mu_3 M G}{r} \quad -(20)$$

Experimentally: $$m \gg \mu_3 \qquad -(21)$$

so $$\mathcal{L} = \frac{1}{2}\mu_3 \dot{r}^2 - \frac{\mu_3 m G}{r} \qquad -(22)$$

In cylindrical planar coordinates:

$$\mathcal{L} = \frac{1}{2}\mu_3\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\mu_3 m G}{r} \qquad -(23)$$

with $$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \qquad -(24)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \qquad -(25)$$

so $$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu_3 r^2}{L^2}F(r) \qquad -(26)$$

and $$r = \frac{\alpha}{1+\epsilon\cos\theta} \qquad -(27)$$

For example:

$$r_1^2 = \left(\frac{\alpha_2}{1+\epsilon_2\cos\theta}\right)^2 + \left(\frac{\alpha_3}{1+\epsilon_3\cos\theta}\right)^2 - 2\left(\frac{\alpha_2\alpha_3\cos\beta}{(1+\epsilon_2\cos\theta)(1+\epsilon_3\cos\theta)}\right) \qquad -(28)$$