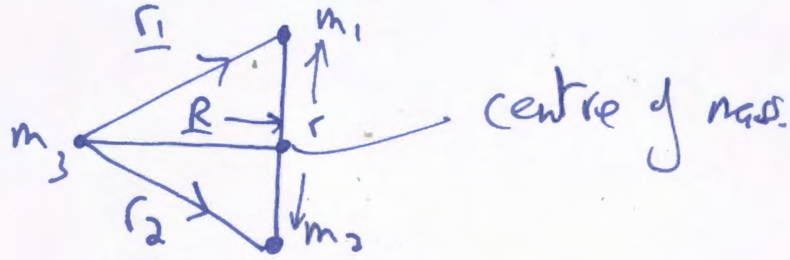
219(5): Statement of the Gravitational Problem

Fig (1)



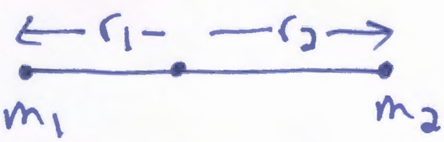
The potential energy is a function only of:

$$r = |\underline{r_1} - \underline{r_2}| \quad -(1)$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m_1 |\dot{\underline{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\underline{r}}_2|^2 - U(r) \quad -(2)$$

In this representation the third mass m_3 is neglected so it is known as the two body problem. The system in Fig (1) is reduced to:



Fig(2)

The translation through space of the centre of mass is ignored, so:

$$m_1 \underline{r}_1 + m_2 \underline{r_2} = \underline{0}. \quad -(3)$$

The vector $\underline{r}$ is defined by:

$$\underline{r} = \underline{r_1} - \underline{r_2}, \quad -(4)$$

so:

$$\underline{r}_1 = \frac{m_2}{m_1 + m_2} \underline{r} \; ; \; \underline{r}_2 = -\frac{m_1}{m_1 + m_2} \underline{r} \quad -(5)$$

The Lagrangian becomes:

2) $$\mathcal{L} = \frac{1}{2}\mu|\dot{\underline{r}}|^2 - U(r) \quad -(6)$$

where the reduced mass is:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad -(7)$$

If the mass of an orbiting object in the solar system is denoted m and the mass of the sun say M then:

$$M \gg m \quad -(8)$$

and

$$\mu \sim m \quad -(9)$$

In cylindrical polar coordinates eq. (6) becomes:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r), \quad -(10)$$

with:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(11)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(12)$$

Robert Hooke inferred that:

$$U(r) = -\frac{mMG}{r} \quad -(13)$$

for a static elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(14)$$

So the two body problem was soluble in the seventeenth century.

3) Here:

$$\alpha = \frac{L^2}{\mu k} \quad ; \quad \epsilon = \left(1 + \frac{2EL^2}{\mu k^2}\right)^{1/2} ; \quad -(15)$$

$$k = mMG. \quad -(16)$$

Eqs. (10) to (12) can be expressed as:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{L^2} r^2 F(r). \quad -(17)$$

The angular momentum is defined by:

$$L = mr^2 \frac{d\theta}{dt}. \quad -(18)$$

The orbital linear velocity is:

$$v = \left(G(m+M)\left(\frac{2}{r} - \frac{1}{a}\right)\right)^{1/2} \quad -(19)$$

so:

$$E = \frac{1}{2} m v^2 - \frac{GmM}{r} = -\frac{GmM}{2a} \quad -(20)$$

The time taken for one orbit is given by Kepler's third law:

$$T^2 = \frac{4\pi^2}{G(m+M)} a^3. \quad -(21)$$

and ~~the~~ angular momentum can be calculated using:

$$\underline{L} = m\, \underline{r} \times \underline{v}. \quad -(22)$$

4) Later it was observed that the elliptical orbit is precessing by an amount:

$$\Delta\theta = 2\pi(x-1) \quad -(23)$$

every orbit. A precessing conical section is described by:

$$\alpha = \frac{r}{1+\epsilon\cos(x\theta)} \quad -(24)$$

The equation of motion (17) then gives:

$$U(r) = -x^2\frac{mMG}{r} + \frac{(x^2-1)L^2}{2mr^2}, \quad -(25)$$

$$F(r) = -x^2\frac{mMG}{r^2} + (x^2-1)\frac{L^2}{mr^3}, \quad -(26)$$

$$v^2 = \left(\frac{L}{m\alpha}\right)^2\left(\frac{2x^2\alpha}{r} + x^2(\epsilon^2-1) - \left(\frac{\alpha}{r}\right)^2(x^2-1)\right) \quad -(27)$$

It is now known that eq. (24) gives a vast array of orbits, even for the two body problem. It is also known that Einsteinian general relativity (EGR) is completely incorrect.

These advances must now be applied to the gravitational N body problem.

5) <u>The Newtonian</u> N <u>Body Problem.</u>

The total energy is:

$$E = T + V = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 - G \sum_{i<j} \frac{m_i m_j}{|\underline{r}_i - \underline{r}_j|} \quad -(28)$$

The potential energy is summed over all pairs of masses. <u>This is</u> the <u>pairwise additive approximation</u>. In general there are $N(N-1)/2$ pairs. If $N = 3$ then:

$$\frac{d^2 \underline{r}_1}{dt^2} = -Gm_2 \frac{(\underline{r}_1 - \underline{r}_2)}{|\underline{r}_1 - \underline{r}_2|^3} - Gm_3 \frac{(\underline{r}_1 - \underline{r}_3)}{|\underline{r}_1 - \underline{r}_3|^3}$$

$$\frac{d^2 \underline{r}_2}{dt^2} = -Gm_3 \frac{(\underline{r}_2 - \underline{r}_3)}{|\underline{r}_2 - \underline{r}_3|^3} - Gm_1 \frac{(\underline{r}_2 - \underline{r}_1)}{|\underline{r}_2 - \underline{r}_1|^3}$$

$$\frac{d^2 \underline{r}_3}{dt^2} = -Gm_1 \frac{(\underline{r}_3 - \underline{r}_1)}{|\underline{r}_3 - \underline{r}_1|^3} - Gm_2 \frac{(\underline{r}_3 - \underline{r}_2)}{|\underline{r}_3 - \underline{r}_2|^3} \quad -(29)$$

In general, eq. (29) is regarded as an unsolved problem. However these three equations can be written using:

$$\underline{R} = \underline{r}_1 - \underline{r}_2 \ ; \ \underline{R}_1 = \underline{r}_1 - \underline{r}_3 \ ; \ \underline{R}_3 = \underline{r}_2 - \underline{r}_3 \quad -(30)$$

6)

$$F_1 = m_1 \frac{d^2 r_1}{dt^2} = -\frac{G m_1 m_2}{R^2} - \frac{G m_1 m_3}{R_1^2} \quad -(31)$$

$$F_2 = m_2 \frac{d^2 r_2}{dt^2} = -\frac{G m_2 m_3}{R_3^2} + \frac{G m_2 m_1}{R^2} \quad -(32)$$

$$F_3 = m_3 \frac{d^2 r_3}{dt^2} = \frac{G m_3 m_1}{R_1^2} + \frac{G m_3 m_2}{R_3^2} \quad -(33)$$

Note that $F_1 + F_2 + F_3 = 0 \quad -(34)$

The lagrangian is:

$$\mathcal{L} = T - U \quad -(35)$$

$$= \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_3 \dot{r}_3^2\right) - U$$

where:

$$U = -\frac{m_1 m_2 G}{R} - \frac{m_1 m_3 G}{R_1} - \frac{m_2 m_3 G}{R_3} \quad -(36)$$

as can be seen from eq. (28), i.e.

$$\boxed{\mathcal{L} = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_3 \dot{r}_3^2\right) - \frac{m_1 m_2 G}{r_1 - r_2} - \frac{m_1 m_3 G}{r_1 - r_3} - \frac{m_2 m_3 G}{r_2 - r_3} \quad -(37)}$$