219(ii): The Four Particle Problem

The Lagrangian in this case is:

$$\mathcal{L} = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2 + m_4|\dot{\underline{r}}_4|^2\right) - \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} - \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} - \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} - \frac{m_1 m_4 G}{|\underline{r}_1 - \underline{r}_4|} - \frac{m_2 m_4 G}{|\underline{r}_2 - \underline{r}_4|} - \frac{m_3 m_4 G}{|\underline{r}_3 - \underline{r}_4|} . \quad -(1)$$

The Lagrangian is written as:

$$\mathcal{L} = \frac{1}{3}\left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6\right) \quad -(2)$$

where:

$$\mathcal{L}_1 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2\right) - \frac{3 m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} \quad -(3)$$

$$\mathcal{L}_2 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{3 m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} \quad -(4)$$

$$\mathcal{L}_3 = \frac{1}{2}\left(m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{3 m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad -(5)$$

$$\mathcal{L}_4 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_4|\dot{\underline{r}}_4|^2\right) - \frac{3 m_1 m_4 G}{|\underline{r}_1 - \underline{r}_4|} \quad -(6)$$

$$\mathcal{L}_5 = \frac{1}{2}\left(m_2|\dot{\underline{r}}_2|^2 + m_4|\dot{\underline{r}}_4|^2\right) - \frac{3 m_2 m_4 G}{|\underline{r}_2 - \underline{r}_4|} \quad -(7)$$

$$\mathcal{L}_6 = \frac{1}{2}\left(m_3|\dot{\underline{r}}_3|^2 + m_4|\dot{\underline{r}}_4|^2\right) - \frac{3 m_3 m_4 G}{|\underline{r}_3 - \underline{r}_4|} \quad -(8)$$

2)

Now use:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0} \quad -(9)$$

$$m_2 \underline{r}_2 + m_3 \underline{r}_3 = \underline{0} \quad -(10)$$

$$m_1 \underline{r}_1 + m_3 \underline{r}_3 = \underline{0} \quad -(11)$$

$$m_1 \underline{r}_1 + m_4 \underline{r}_4 = \underline{0} \quad -(12)$$

$$m_2 \underline{r}_2 + m_4 \underline{r}_4 = \underline{0} \quad -(13)$$

$$m_3 \underline{r}_3 + m_4 \underline{r}_4 = \underline{0} \quad -(14)$$

which calculate the centres of mass of each interacting pairs of particles. Next define:

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \ \mu_2 = \frac{m_1 m_3}{m_1 + m_3}, \ \mu_3 = \frac{m_2 m_3}{m_2 + m_3},$$

$$\mu_4 = \frac{m_1 m_4}{m_1 + m_4}, \ \mu_5 = \frac{m_2 m_4}{m_2 + m_4}, \ \mu_6 = \frac{m_3 m_4}{m_3 + m_4}. \quad -(15)$$

Also define:

$$k_1 = 3 m_1 m_2 G, \ k_2 = 3 m_1 m_3 G, \ k_3 = 3 m_2 m_3 G,$$

$$k_4 = 3 m_1 m_4 G, \ k_5 = 3 m_2 m_4 G, \ k_6 = 3 m_3 m_4 G. \quad -(16)$$

As in note 219(10) the six Lagrangians (3) to (8) reduce to the general format:

$$\mathcal{L} = \frac{1}{2} \mu |\underline{\dot{r}}|^2 - U(r) \quad -(17)$$

The Euler Lagrange equations are:

3)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(18)$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(19)$$

i.e

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L^2} F(r) \quad -(20)$$

where

$$F(r) = -\partial U(r) / \partial r. \quad -(21)$$

Therefore there are six orbits for each pair of particles:

$$\boxed{r_i = \frac{\alpha_i}{1 + \epsilon_i \cos\theta}, \quad i = 1, \ldots, 6} \quad -(22)$$

where

$$\alpha_i = \frac{L_i^2}{\mu_i k_i}, \quad i = 1, \ldots, 6 \quad -(23)$$

$$\epsilon_i = \left(1 + \frac{2E_i L_i^2}{\mu_i k_i^2}\right)^{1/2}, \quad i = 1, \ldots, 6 \quad -(24)$$

with:

$$L_i = \mu_i r_i^2 \dot{\theta}, \quad -(25)$$

$$i = 1, \ldots, 6$$

The orbits are constrained by the fact that θ has no index (no subscript i), so:

$$\cos\theta = \frac{1}{\epsilon_i}\left(\frac{d_i}{r_i} - 1\right), \quad -(26)$$

$$i = 1, \ldots, 6,$$

giving:

$$r_{i+1} = d_{i+1}\left(1 - \frac{\epsilon_{i+1}}{\epsilon_i}\left(\frac{d_i - r_i}{r_i}\right)\right) \quad -(27)$$

$$i = 1, \ldots, 6. \quad -(28)$$

Therefore:

$$r_{i+2} = d_{i+2}\left(1 - \frac{\epsilon_{i+2}}{\epsilon_{i+1}}\left(\frac{d_{i+1} - r_{i+1}}{r_{i+1}}\right)\right)$$

with r_{i+1} given by eq. (27).

This procedure can be extended to N pair ~~particles~~:

$$i = 1, \ldots, N.$$

Example: 3 Particles.

In this case: $i = 1, 2, 3.$ $-(29)$

So:

$$r_3 = d_3\left(1 - \frac{\epsilon_3}{\epsilon_2}\left(\frac{d_2 - r_2}{r_2}\right)\right), \quad -(30)$$

where:

5)
$$r_2 = \alpha_2\left(1 - \frac{\epsilon_2}{\epsilon_1}\left(\frac{\alpha_1 - r_1}{r_1}\right)\right) \quad -(31)$$

and
$$r_1 = \frac{\alpha_1}{1+\epsilon_1\cos\theta}. \quad -(32)$$

In addition:
$$r_2 = \frac{\alpha_2}{1+\epsilon_2\cos\theta}, \quad -(33)$$
$$r_3 = \frac{\alpha_3}{1+\epsilon_3\cos\theta}, \quad -(34)$$
$$\frac{1}{\epsilon_1}\left(\frac{\alpha_1}{r_1} - 1\right) = \frac{1}{\epsilon_2}\left(\frac{\alpha_2}{r_2} - 1\right) \quad -(35)$$
$$\frac{1}{\epsilon_1}\left(\frac{\alpha_1}{r_1} - 1\right) = \frac{1}{\epsilon_3}\left(\frac{\alpha_3}{r_3} - 1\right) \quad -(36)$$
$$\frac{1}{\epsilon_2}\left(\frac{\alpha_2}{r_2} - 1\right) = \frac{1}{\epsilon_3}\left(\frac{\alpha_3}{r_3} - 1\right) \quad -(37)$$

Eqs. (32) to (37) are six equations in nine unknowns.
To find three more equations use:
$$\alpha_i = \frac{L_i^2}{\mu_i k_i} = \frac{\mu_i r_i^2}{k_i}\dot{\theta} \quad -(38)$$
$$i = 1, 2, 3$$

Therefore:

6)

$$\frac{d_1}{d_2} = \frac{\mu_1 r_1^2}{k_1} \frac{k_2}{\mu_2 r_2^2}$$

$$= \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{k_2}{k_1}\right)\left(\frac{r_1}{r_2}\right)^2 \quad -(39)$$

Here:

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 = \frac{m_1 m_3}{m_1 + m_3}, \quad -(40)$$

$$k_1 = 3 m_1 m_2 G, \quad k_2 = 3 m_1 m_3 G.$$

So

$$\frac{d_1}{d_2} = \left(\frac{m_1 + m_3}{m_1 + m_2}\right)\left(\frac{r_1}{r_2}\right)^2 \quad -(41)$$

$$\frac{d_1}{d_3} = \left(\frac{m_2 + m_3}{m_1 + m_2}\right)\left(\frac{r_1}{r_3}\right)^2 \quad -(42).$$

$$\frac{d_2}{d_3} = \left(\frac{m_2 + m_3}{m_1 + m_3}\right)\left(\frac{r_2}{r_3}\right)^2 \quad -(43).$$

This gives a set of nine equations, (32), (33), (34), (35), (36), (37), (41), (42) and (43) in nine unknowns: $r_1, r_2, r_3, d_1, d_2, d_3, \epsilon_1, \epsilon_2, \epsilon_3$.

Conclusion

The problem is soluble analytically.