19(10): Analytical Solution of the N Particle Problem.

For the three particle problem:

$$\mathcal{L} = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_3 \dot{r}_3^2\right) - \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} - \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} - \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad -(1)$$

$$= \frac{1}{2}\left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3\right), \quad -(2)$$

where:

$$\mathcal{L}_1 = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2\right) - \frac{2 m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|}, \quad -(3)$$

$$\mathcal{L}_2 = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_3 \dot{r}_3^2\right) - \frac{2 m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|}, \quad -(4)$$

$$\mathcal{L}_3 = \frac{1}{2}\left(m_1 \dot{r}_2^2 + m_3 \dot{r}_3^2\right) - \frac{2 m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|}. \quad -(5)$$

The 3 particle lagrangian $\mathcal{L}$ has been reduced to the sum of three 2 particle lagrangians (3) to (5). Similarly it can be shown that the 4 particle lagrangian can be reduced to the sum of six 2 particle lagrangian. In general, the n particle lagrangian can be reduced to the sum of $n!/((n-2)!\,2!)$ 2 particle lagrangians.

Each of the lagrangians in eqs. (3) and (4) is analytically soluble for motion in a plane, so the solution for eq. (1) is the combined motion of eqs. (3) to (5).

2) The Lagrangians (3) to (5) can be written as:

$$\mathcal{L}_1 = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2\right) - U(R_1) \quad -(6)$$

$$\mathcal{L}_2 = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_3 \dot{r}_3^2\right) - U(R_2) \quad -(7)$$

$$\mathcal{L}_3 = \frac{1}{2}\left(m_1 \dot{r}_2^2 + m_3 \dot{r}_3^2\right) - U(R_3) \quad -(8)$$

where:

$$\underline{R}_1 = \underline{r}_1 - \underline{r}_2 \quad -(9)$$

$$\underline{R}_2 = \underline{r}_1 - \underline{r}_3 \quad -(10)$$

$$\underline{R}_3 = \underline{r}_2 - \underline{r}_3 . \quad -(11)$$

In eq. (6) define: $m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0} \quad -(12)$ so that the origin of the coordinate system is the centre of mass. Similarly:

$$m_1 \underline{r}_1 + m_3 \underline{r}_3 = \underline{0} \quad -(13)$$

and

$$m_2 \underline{r}_2 + m_3 \underline{r}_3 = \underline{0} . \quad -(14)$$

With these definitions:

$$\boxed{\mathcal{L}_i = \frac{1}{2}\mu_i |\dot{\underline{R}}_i|^2 - U_i, \quad i = 1, 2, 3} \quad -(15)$$

Here:

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 = \frac{m_1 m_3}{m_1 + m_3}, \quad \mu_3 = \frac{m_2 m_3}{m_2 + m_3} \quad -(16)$$

It is convenient to change the notation in eq. (15) to:

$$\mathcal{L}_i = \frac{1}{2}\mu_i |\underline{\dot{r}}_i|^2 - U_i, \quad -(17)$$

which is a lagrangian of the type:

$$\mathcal{L} = \frac{1}{2}\mu |\underline{\dot{r}}|^2 - U(r). \quad -(18)$$

In eq (18):

$$U(r) = -\frac{2mMG}{r} \quad -(19)$$

$$:= -\frac{k}{r}$$

where

$$k = 2mMG. \quad -(20)$$

Thus:

$$k_1 = 2m_1m_2G, \quad k_2 = 2m_1m_3G, \quad k_3 = 2m_2m_3G \quad -(21).$$

In cylindrical polar coordinates in a plane:

$$\underline{r} = r\,\underline{e}_r \quad -(22)$$

$$\underline{\dot{r}} = \frac{d}{dt}(r\,\underline{e}_r) \quad -(23)$$

The radial unit vector is defined by:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta, \quad -(24)$$

and the other unit vector ~~are~~ is:

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta. \quad -(25)$$

In eq. (23):

$$\underline{\dot{r}} = \dot{r}\,\underline{e}_r + r\,\underline{\dot{e}}_r . \qquad -(26)$$

For each particle $i = 1, 2, 3$:

$$\boxed{\begin{aligned} \underline{r}_i &= r_i\,\underline{e}_r \\ \underline{\dot{r}}_i &= \dot{r}_i\,\underline{e}_r + r_i\,\underline{\dot{e}}_r \end{aligned}} \qquad \begin{aligned} &-(27) \\ &-(28) \end{aligned}$$

Note carefully that $\underline{\dot{e}}_r$ does not depend on i, because it is defined by:

$$\underline{\dot{e}}_r = \frac{d\underline{e}_r}{dt} \qquad -(29)$$

and $\underline{e}_r$ is a unit vector. So the Euler Lagrange equations for the Lagrangian (17) are:

$$\frac{\partial \mathcal{L}_i}{\partial r_i} = \frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{r}_i} \qquad -(30)$$

$$i = 1, 2, 3.$$

From eqs. (24) and (25):

$$\underline{\dot{e}}_r = \dot{\theta}\,\underline{e}_\theta \qquad -(30)$$

and $\dot{\theta}$ does not have an index i.

So the lagrangian (17) is:

$$\mathcal{L}_i = \frac{1}{2}\mu_i\left(\dot{r}_i^2 + \dot{\theta}^2 r_i^2\right) - U(r_i) \quad -(31)$$

with

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(32)$$

From eqs. (30) and (32):

$$\left.\begin{aligned} \frac{\partial \mathcal{L}_i}{\partial r_i} &= \frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{r}_i}, \\ \frac{\partial \mathcal{L}}{\partial \theta} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\ i &= 1, 2, 3 \end{aligned}\right\} \quad -(33)$$

These can be rewritten as:

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r_i}\right) + \frac{1}{r_i} = -\frac{\mu_i r_i^2}{L_i^2} F_i(r)} \quad -(34)$$

where

$$L_i = \mu_i r_i^2 \frac{d\theta}{dt}, \quad -(35)$$

$$F_i = -\partial U_i / \partial r. \quad -(36)$$

The solution of eq. (34) is given by three orbits:

6)

$$r_i = \frac{\alpha_i}{1+\epsilon_i \cos\theta} \quad -(37)$$

where:

$$\alpha_i = \frac{L_i^2}{\mu_i k_i} \quad -(38)$$

$$\epsilon_i = \left(1 + \frac{2E_i L_i^2}{\mu_i k_i^2}\right)^{1/2}, \quad -(39)$$

$$\left.\begin{aligned} k_1 &= 2m_1 m_2 G \\ k_2 &= 2m_1 m_3 G \\ k_3 &= 2m_2 m_3 G \end{aligned}\right\} \quad -(40)$$

These orbits are linked together by:

$$\cos\theta = \frac{1}{\epsilon_1}\left(\frac{\alpha_1}{r_1} - 1\right) = \frac{1}{\epsilon_2}\left(\frac{\alpha_2}{r_2} - 1\right) = \frac{1}{\epsilon_3}\left(\frac{\alpha_3}{r_3} - 1\right) \quad -(41)$$

i.e. for example r_2 is given by:

$$\frac{1}{\epsilon_2}\left(\frac{\alpha_2 - r_2}{r_2}\right) = \frac{1}{\epsilon_1}\left(\frac{\alpha_1 - r_1}{r_1}\right) \quad -(42)$$

giving

$$\boxed{r_2 = \alpha_2 \left(1 - \frac{\epsilon_2}{\epsilon_1}\left(\frac{\alpha_1 - r_1}{r_1}\right)\right)},$$

where

$$r_1 = \frac{\alpha_1}{1+\epsilon_1 \cos\theta} \quad -(43)$$