219(9): Simplest Solution of the Three Particle Problem

Consider two planets of mass m_1 and m_2 orbiting the sun of mass M in a plane. The centre of mass of m_1 and m_2 is determined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0}. \quad -(1)$$

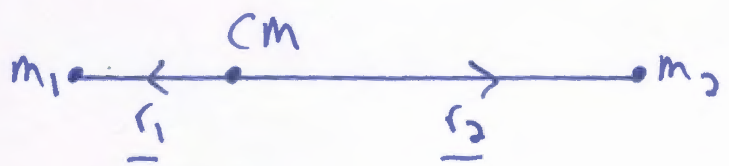
The two masses are separated by:

$$\underline{r} = \underline{r}_1 - \underline{r}_2 \quad -(2)$$

and the lagrangian for m_1 and m_2 is:

$$\mathcal{L} = \frac{1}{2} m_1 |\dot{\underline{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\underline{r}}_2|^2 - U(r) \quad -(3)$$

Fig(1)



From eqs. (1) and (2):

$$\underline{r}_1 = \frac{m_2}{m_1 + m_2} \underline{r}, \quad \underline{r}_2 = -\frac{m_1}{m_1 + m_2} \underline{r} \quad -(4)$$

From eqs. (3) and (4):

$$\mathcal{L} = \frac{1}{2} \mu |\dot{\underline{r}}|^2 - U(r) \quad -(5)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad -(6)$$

Therefore the lagrangian (5) describes the motion of the reduced mass μ and the distance $\underline{r}$ between m_1 and m_2.

2)

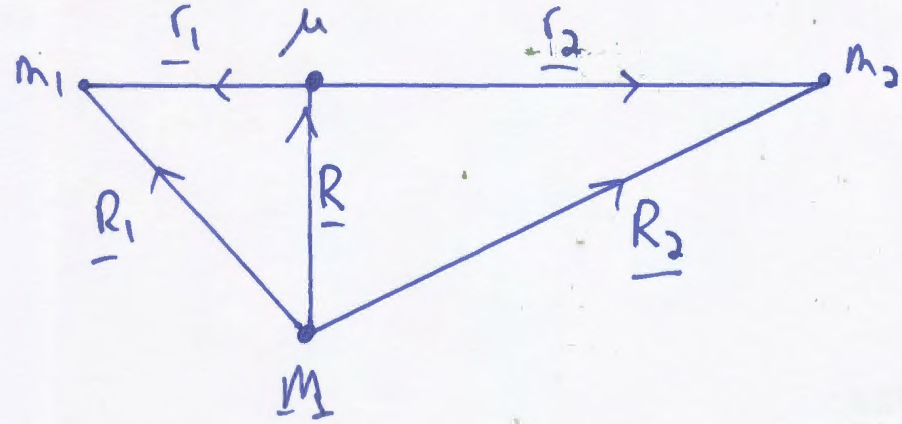


Fig.(2) Fig(2)

Similarly the motion of μ around $\underline{M}$ is described by:

$$\mathcal{L}_1 = \frac{1}{2}\mu_1 |\dot{\underline{R}}|^2 - u(R) \quad -(7)$$

where

$$\mu_1 = \frac{\mu M}{\mu + M}. \quad -(8)$$

The orbit of μ around M is:

$$R = \frac{\alpha}{1+\epsilon\cos(x\theta)}, \quad -(9)$$

with conserved angular momentum:

$$L = \mu_1 R^2 \frac{d\theta}{dt}. \quad -(10)$$

Here:

$$\left.\alpha = \frac{L^2}{\mu_1 k}, \quad \epsilon = \left(1 + \frac{2EL^2}{\mu_1 k^2}\right)^{1/2}, \right\} \quad -(11)$$

$$k = \mu M G$$

The distances R_1 and R_2 between planets m_1 and m_2 are given by:

$$\underline{R_1} = \underline{R} + \underline{r_1} \quad -(12)$$

$$\underline{R_2} = \underline{R} + \underline{r_2} \quad -(13)$$

i.e.

$$\underline{R_1} = \underline{R} + \left(\frac{m_2}{m_1+m_2}\right)\underline{r} \quad -(14)$$

$$\underline{R}_2 = \underline{R} - \left(\frac{m_1}{m_1+m_2}\right)\underline{r} \quad -(15)$$

The orbit of r is given by:

$$r = \frac{\alpha_1}{1+\epsilon_1 \cos(x_1 \theta)} \quad -(16)$$

with angular momentum:

$$L_1 = \mu r^2 \frac{d\theta}{dt}, \quad -(17)$$

$$\alpha_1 = \frac{L_1^2}{\mu k_1}, \quad \epsilon_1 = \left(1 + \frac{2E_1 L_1^2}{\mu k_1^2}\right) \quad -(18)$$

$$k_1 = m_1 m_2 G$$

From eqns (14) and (15):

$$\boxed{\underline{R_1} - \underline{R_2} = \left(\frac{m_2-m_1}{m_1+m_2}\right)\underline{r}} \quad -(19)$$

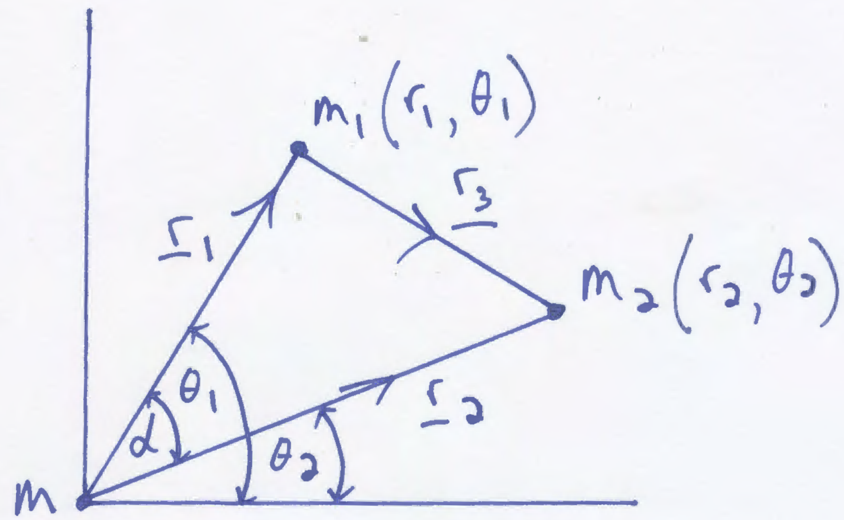
so:

$$r = \left(\frac{m_1+m_2}{m_2-m_1}\right)\left|\underline{R}_1 - \underline{R_2}\right| \quad -(20)$$

219(10): Alignment of Two Planets with the Sun.

In general the geometry is sketched in Fig. (1).

Fig(1)



It is described by:

$$\alpha = \theta_1 - \theta_2, \quad -(1)$$

$$\underline{r_1} = \underline{r_2} - \underline{r_3}. \quad -(2)$$

So:

$$r_1^2 = (\underline{r_2} - \underline{r_3}) \cdot (\underline{r_2} - \underline{r_3}) \quad -(3)$$

$$= r_2^2 + r_3^2 - 2 r_2 r_3 \cos(\theta_1 - \theta_2).$$

Here:

$$r_1 = \frac{\alpha_1}{1 + \epsilon_1 \cos(x_1 \theta_1)}, \quad r_2 = \frac{\alpha_2}{1 + \epsilon_2 \cos(x_2 \theta_2)} \quad -(4)$$

From eq. (3):

$$r_3^2 + A r_3 + B = 0 \quad -(5)$$

where:

2)

$$A = -2r_2\cos(\theta_1-\theta_2), \quad B = r_2^2 - r_1^2 \quad -(6)$$

so

$$r_3 = \frac{1}{2}\left(-A \pm \left(A^2-4B\right)^{1/2}\right) \quad -(7)$$

$$= \frac{1}{2}\left(2r_2\cos(\theta_1-\theta_2) \pm \left(4r_2^2\cos^2(\theta_1-\theta_2) - 4(r_2^2-r_1^2)\right)^{1/2}\right)$$

i.e.

$$r_3 = r_2\cos(\theta_1-\theta_2) \pm \left(r_1^2 - r_2^2\left(1-\cos^2(\theta_1-\theta_2)\right)\right)^{1/2} \quad -(8)$$

If m, m_1 and m_2 are in alignment:

$$\theta_1 = \theta_2 \quad -(9)$$

so

$$\boxed{r_3 = \frac{\alpha_2}{1+\epsilon_1\cos(x_1\theta_1)} - \frac{\alpha_1}{1+\epsilon_2\cos(x_2\theta_2)}} \quad -(10)$$

If there were no interaction between m_1 and m_2 then r_3 is given by eq. (8). However, in the three particle problem:

$$r_3 = \frac{\alpha_3}{1+\epsilon_3\cos(x_3\theta_3)} \quad -(11)$$

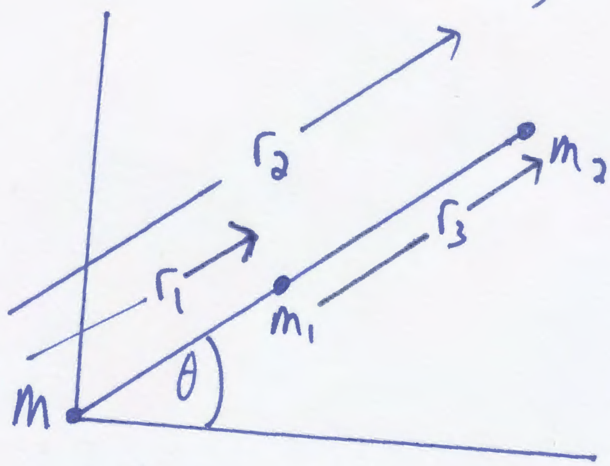
Therefore from eqs. (3) and (11) it is possible to derive a relation between θ_1, θ_2 and θ_3.

In the case of alignment eq. (10) can be written as:

$$\theta_1 = \theta_2 = \theta \quad -(12)$$

so

$$r_3 = \frac{\alpha_2}{1 + \epsilon_1 \cos(x_1 \theta)} - \frac{\alpha_1}{1 + \epsilon_2 \cos(x_2 \theta)} \quad -(13)$$



Fig(2)

If there is no gravitational interaction between m_1 and m_2 then r_3 can be plotted against θ as in eq. (13). If there is gravitational interaction then:

$$r_3 = \frac{\alpha_3}{1 + \epsilon_3 \cos(x_3 \theta_3)} \quad -(14)$$

so:

$$\frac{\alpha_3}{1 + \epsilon_3 \cos(x_3 \theta_3)} = \frac{\alpha_2}{1 + \epsilon_1 \cos(x_1 \theta)} - \frac{\alpha_1}{1 + \epsilon_2 \cos(x_2 \theta)} \quad -(15)$$

and θ_3 can be plotted against θ.

4) In the Newtonian limit:

$$\frac{\alpha_3}{1+\epsilon_3 \cos\theta_3} = \frac{\alpha_2}{1+\epsilon_1 \cos\theta} - \frac{\alpha_1}{1+\epsilon_2 \cos\theta} \quad -(16)$$

$$= \frac{\alpha_2\left(1+\epsilon_2 \cos\theta\right) - \alpha_1\left(1+\epsilon_1 \cos\theta\right)}{\left(1+\epsilon_1 \cos\theta\right)\left(1+\epsilon_2 \cos\theta\right)}$$

So:

$$1+\epsilon_3 \cos\theta_3 = \frac{\alpha_3\left(1+\epsilon_1 \cos\theta\right)\left(1+\epsilon_2 \cos\theta\right)}{\alpha_2\left(1+\epsilon_2 \cos\theta\right) - \alpha_1\left(1+\epsilon_1 \cos\theta\right)} \quad -(17)$$

$$\boxed{\theta_3 = \cos^{-1}\left[\frac{1}{\epsilon_3}\left(\frac{\alpha_3\left(1+\epsilon_1 \cos\theta\right)\left(1+\epsilon_2 \cos\theta\right)}{\alpha_2\left(1+\epsilon_2 \cos\theta\right) - \alpha_1\left(1+\epsilon_1 \cos\theta\right)}\right) - 1\right]}$$

$-(18)$