219(12): Proof that the Quantity $\theta$ has no Index.

Consider for simplicity the two particle problem. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(m_1 |\dot{\underline{r}}_1|^2 + m_2 |\dot{\underline{r}}_2|^2\right) - u(r) \quad -(1)$$

$$= \frac{1}{2}\mu |\dot{\underline{r}}|^2 - u(r) \; ; \; \mu = m_1 m_2 / (m_1 + m_2).$$

Consider the centre of mass defined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 \quad -(2)$$
$$= \underline{0}$$

and

$$\underline{r} = \underline{r}_1 - \underline{r}_2 . \quad -(3)$$

From eq. (2):

$$\underline{r}_1 = \frac{m_2}{m_1 + m_2} \underline{r} \; ; \; \underline{r}_2 = -\frac{m_1}{m_1 + m_2} \underline{r} . \quad -(4)$$

It follows that

$$r_1^2 = \left(\frac{m_2}{m_1 + m_2}\right)^2 r^2 ; \; r_2^2 = \left(\frac{m_1}{m_1 + m_2}\right)^2 r^2 ; \quad -(5)$$

and

$$|\dot{\underline{r}}_1|^2 = \left(\frac{m_2}{m_1 + m_2}\right)^2 |\dot{\underline{r}}|^2 \quad -(6)$$

$$|\dot{\underline{r}}_2|^2 = \left(\frac{m_1}{m_1 + m_2}\right)^2 |\dot{\underline{r}}_1|^2 \quad -(7)$$

Therefore:

$$\mathcal{L} = \frac{1}{2}\left(\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2}\right) |\dot{\underline{r}}|^2 - u(r)$$

$$= \frac{1}{2}\left(\frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2}\right) |\dot{\underline{r}}|^2 - u(r)$$

2) $$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)|\dot{\underline{r}}|^2 - U(r) \quad (8)$$

where the reduced mass is:

$$\mu = \frac{m_1 m_2}{m_1+m_2} \quad (9)$$

In plane cylindrical coordinates:

$$|\dot{\underline{r}}_1|^2 = \dot{r}_1^2 + \dot{\theta}^2 r_1^2 \quad (10)$$

$$|\dot{\underline{r}}_2|^2 = \dot{r}_2^2 + \dot{\theta}^2 r_2^2 \quad (11)$$

where:

$$r_1^2 = \left(\frac{m_2}{m_1+m_2}\right)^2 r^2 \quad (12)$$

$$\dot{r}_1^2 = \left(\frac{m_2}{m_1+m_2}\right)^2 \dot{r}^2 \quad (13)$$

$$r_2^2 = \left(\frac{m_1}{m_1+m_2}\right)^2 r^2 \quad (14)$$

$$\dot{r}_2^2 = \left(\frac{m_1}{m_1+m_2}\right)^2 \dot{r}^2. \quad (15)$$

Also:

$$|\dot{\underline{r}}|^2 = \dot{r}^2 + \dot{\theta}^2 r^2 \quad (16)$$

It follows from eqs. (10), (11) and (16) that $\theta$ does not take an index subscript $i$,

Q.E.D. This result provides an important constraint on the dynamics of three or more interacting particles.

3)

The index appears in:

$$\underline{r}_i = r_i \, \underline{e}_r \quad -(17)$$

where $\underline{e}_r$ is the radial unit vector of the plane cylindrical coordinate system:

$$\underline{e}_r = \underline{i} \cos\theta + \underline{j} \sin\theta \quad -(18)$$

in which $\theta$ depends on time. The angular unit vector is:

$$\underline{e}_\theta = -\underline{i} \sin\theta + \underline{j} \cos\theta. \quad -(19)$$

Therefore:

$$\dot{\underline{e}}_r = \frac{d\underline{e}_r}{dt} = \dot{\theta} \, \underline{e}_\theta , \quad -(20)$$

and

$$\frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{d}{dt}\left(r \underline{e}_r\right)$$

$$= \frac{dr}{dt} \underline{e}_r + r \frac{d\underline{e}_r}{dt} \quad -(21)$$

$$= \dot{r} \, \underline{e}_r + \dot{\theta} \, r \, \underline{e}_\theta$$

It follows that:

$$\boxed{\dot{\underline{r}}_i = \dot{r}_i \underline{e}_r + \dot{\theta} \, r_i \, \underline{e}_\theta} \quad -(22)$$

By definition:

$$X_i = r_i \cos\theta \quad -(23)$$

$$Y_i = r_i \sin\theta \quad -(24)$$

4)

and $\underline{r}_i = X_i \underline{i} + Y_i \underline{j}$ – (25)

so: $r_i = (X_i^2 + Y_i^2)^{1/2}$ – (26)

The Cartesian unit vectors $\underline{i}$ and $\underline{j}$ do not take an index subscript $i$.

For the three particle gravitational problem this analysis leads to the result:

$$\boxed{\cos\theta = \frac{1}{\epsilon_1}\left(\frac{d_1}{r_1} - 1\right) = \frac{1}{\epsilon_2}\left(\frac{d_2}{r_2} - 1\right) = \frac{1}{\epsilon_3}\left(\frac{d_3}{r_3} - 1\right)} \quad -(27)$$

which is a constraint equation of key importance. Eq. (27) follows from the fact that $\theta$ has no index, and is derived from the three inter-related two particle orbits:

$$r_i = \frac{d_i}{1 + \epsilon_i \cos\theta} \quad -(28)$$

$$i = 1, 2, 3$$

in the Newtonian limit. For precessional orbits:

$$\boxed{r_i = \frac{d_i}{1 + \epsilon_i \cos(x_i \theta)}, \quad i = 1, 2, 3} \quad -(29)$$

5) In the Newtonian limit eq. (27) is an analytical solution of the 3 particle problem. For the N particle problem:

$$\cos\theta = \frac{1}{\epsilon_1}\left(\frac{\alpha_1}{r_1} - 1\right) = \cdots = \frac{1}{\epsilon_i}\left(\frac{\alpha_i}{r_i} - 1\right), \quad -(30)$$

$$i = 1, \ldots, N$$

So we reach the important conclusion that the N particle problem is soluble analytically. This seems to be the first general solution discovered in over 350 years.

For precessing orbits each with an $x$ factor $x_i$, $i = 1, \ldots, N$, the constraint (27) becomes:

$$\theta = \frac{1}{x_1}\cos^{-1}\left(\frac{1}{\epsilon_1}\left(\frac{\alpha_1}{r_1} - 1\right)\right) = \cdots = \frac{1}{x_i}\cos^{-1}\left(\frac{1}{\epsilon_i}\left(\frac{\alpha_i}{r_i} - 1\right)\right) \quad -(31)$$

$$i = 1, \ldots, N.$$

The study of the precessing N particle problem is of fundamental importance.